Selective Sensing and Transmission for Multi-Channel Cognitive Radio Networks

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Abstract—To maximize SU's temporal channel utilization while limiting its interference to PUs, a selective sensing and selective access (SS-SA) strategy for one slotted SU overlaying a nontime-slotted ON/OFF continuous time Markov chain (CTMC) modeled multi-channel primary network is proposed. Under the proposed selective sensing strategy, each channel will be detected approximate periodically with different periods according to the parameter T_c , which reflects the maximal period that each channel should be probed. Once the spectrum hole is found, if the sensing period is suitable, the SU could continuously access the channel until it sense this channel next time. Numerical simulations illustrate that T_c is a valid measurement to indicate how often the channel should be sensed, and with the SS-SA strategy, SU can effectively utilize the channels and consume less energy and time for sensing than two reference strategies.

I. INTRODUCTION

FCC's report indicates that the current spectrum management policy has resulted in an under-utilized spectrum [1]. To improve spectrum utilization, cognitive radio (CR) [2] is proposed. Its basic idea is to allow secondary user (SU) to search for and utilize instantaneous spectrum opportunities left by primary user, while limiting its interference to PU.

To utilize spectrum opportunities, the SU should first model PU's behavior. There are mainly two models, namely, discretetime and continuous-time models. In discrete-time model, both PU and SU are time-slotted. In [3], a dynamic programming approach to search the optimal sensing order is proposed. In [4], an opportunistic MAC protocol with random and negotiation-based sensing for ad-hoc networks is proposed. In [5], the authors derive the optimal spectrum sensing and access strategies under the formulation of POMDP. For this model, since the synchronization of all PUs and SUs is necessary, it causes more overhead and time offset may be fatal for SU's MAC strategy. In continuous-time model, the PUs are nonslotted but the SUs are still slotted mostly. Since PU's state may change at any time, this model is more difficult to analyze. The authors of [6] derive the optimal access strategy with periodic sensing for one SU overlapping a CTMC modeled multi-channel primary network. In [7], [8], the optimal access strategy with fully sensing strategy is obtained. However, none of these works investigate the magnitude of sensing period. In [9]–[11], the optimal sensing period is derived for the simplest single-channel model.

In this paper, we consider a CR network which has multiple channels available for transmissions by primary and secondary users. We assume each channel is assigned to an independent PU and the time behavior of each channel is modeled by a two-state (ON/OFF) first-order CTMC model. Meanwhile, one slotted SU can access all of these channels simultaneously. Since generally how often each channel should be sensed is distinct and it will take more energy and time to sense all channels simultaneously, SU could only sense part of the channels. Thus, SU could save more energy and time for transmission. We assume that SU senses only one channel in each slot (the proposed sensing strategy can be easily be generalized to the case when SU probes multiple channels each time). Therefore, in each slot, SU decides which channel should be sensed first and in which channels to transmit. Furthermore, the magnitude of sensing period is also considered.

The main contributions of this paper are as follows. To maximize SU's temporal channel utilization while limiting its interference to PUs, we propose a selective sensing and selective access (SS-SA) strategy for one slotted SU overlaying a non-time-slotted ON/OFF CTMC modeled multi-channel primary network. With SS strategy, each channel will be detected approximate periodically with different period according to $T_{\rm c}$. The parameter $T_{\rm c}$, which is related to channel's characteristic parameters and interference tolerances, is a valid measurement to indicate how often each channel should be sensed. If sensing period is suitable, the SA strategy can be regarded as greedy access strategy. With SS-SA strategy, SU can effectively utilize these channels and adopt larger sensing period than reference strategies, which means SU could consume less energy and time for sensing. Furthermore, SS-SA strategy is simple and easy to implement.

The rest of the paper is organized as follows. After introducing the system model and problem formulation in Section II,

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the PS-SA and SS-SA strategies are studied in Section III and IV, respectively. In Section V, the simulation results are present and discussed. Finally, conclusions are stated in Section VI.

II. SYSTEM MODEL AND PROBLEM FORMULATION

A. System Model

We consider a multi-channel CR network which has multiple channels available for primary and secondary users. Particularly, there are N channels and each channel is assigned to an independent PU respectively. We assume there is only one SU who can access these channels simultaneously, and its transmission on one channel will not interfere with other channels. To achieve this, we can simply adopt D-OFDM as the PHY technique with a single radio equipment [12]. We assume that all PUs exhibit a non-time-slotted behavior and their activities are independent, while SU employs a time slotted protocol with period $T_s > 0$. Furthermore, SU adopts a "Listen-Before-Talk" strategy. Since generally how often each channel should be sensed is distinct and it will take more energy and time to sense all channels simultaneously, SU could sense part of these channels each time. For the convenience of analysis, we assume SU senses only one channel it each slot. Therefore, SU needs a sensing and access strategy to decide which channel should be sensed first and in which channels to transmit, which is the main objective of this paper. Besides, for ease of analysis, we assume perfect sensing and sensing time is short enough to be ignored. The time behaviors of primary and secondary users are shown in Fig. 1.

B. The Channel Model

The channel's time behavior is modeled as a two-state (ON/OFF) first-order CTMC. This CTMC model has been considered in many spectrum sharing studies including theoretical analysis and hardware tests [6]–[8], [13]–[15]. Based on stochastic theory [16], for any channel *i*, holding times in ON and OFF states are exponentially distributed with parameters μ_i and λ_i , respectively, and the transition matrix is

$$\mathbf{P}(\tau) = \frac{1}{\lambda_i + \mu_i} \begin{bmatrix} \mu_i + \lambda_i e^{-(\lambda_i + \mu_i)\tau} & \lambda_i - \lambda_i e^{-(\lambda_i + \mu_i)\tau} \\ \mu_i - \mu_i e^{-(\lambda_i + \mu_i)\tau} & \lambda_i + \mu_i e^{-(\lambda_i + \mu_i)\tau} \end{bmatrix}.$$
(1)

If sensing result is OFF at time t_0 , then the probability of channel state being ON at time $t_0 + \tau$ is $\frac{1}{\lambda_i + \mu_i} (\lambda_i - \lambda_i e^{-(\lambda_i + \mu_i)\tau})$.

C. Problem Formulation

We focus on maximizing SU's total channel utilization while limiting its interference to PUs. Particularly, the interference between PU and SU is modeled by average temporal overlap, namely, the interference time divided by total time. Mathematically, the interference I_i between SU and PU_i is

$$I_i = \lim_{T \to \infty} \frac{\int_0^T \mathbb{1}\left\{A_i(\tau) \cap B_i(\tau)\right\} d\tau}{T}$$
(2)

where $1\{\cdot\}$ is the indicator function of the event enclosed in the brackets; $A_i(\tau)$ and $B_i(\tau)$ denote the event that PU_i and SU access channel *i* at time τ , respectively.



Fig. 1. The time behavior of primary and secondary users (4 channels).

The channel utilization is defined as SU's temporal utilization ratio (i.e., the transmission time divided by total time). Mathematically, SU's channel utilization U_i on channel *i* is

$$U_{i} = \lim_{T \to \infty} \frac{\int_{0}^{T} 1\{B_{i}(\tau)\} d\tau}{T}.$$
 (3)

Therefore, we focus on the problem P:

$$\max \qquad \sum_{i=1}^{N} U_i \qquad (4)$$

s.t.
$$I_i \le C_i, \ i = 1, \cdots, N$$
 (5)

where $C_i \in [0, 1]$ is the maximum interference level tolerable by primary user *i*. Generally, C_i is very small.

Since SU's sensing and access strategy and sensing period T_s jointly affect SU's interference to PUs and channel utilization, we will study SU's sensing and access strategy and the effect of sensing period T_s .

III. PERIODIC SENSING AND SELECTIVE ACCESS

In this section, we study the optimal access strategy with periodic sensing (PS). Since SU detects these channels one by one, each channel is also probed periodically with period $T_P = NT_s$.

A. Sub-problems of the Original Problem P

We first simplify problem **P** to facilitate analysis. From the perspective of time, in each slot, SU should decide how to access N channels. Since the interferences between SU and each PU don't interact with each other, problem **P** can be decoupled into N independent sub-problems **P**_i:

$$\max \quad U_i \tag{6}$$

s.t.
$$I_i \le C_i$$
. (7)

That is to maximize SU's channel utilization on channel i while limiting its interference to PU_i .

Therefore, from the perspective of each channel, SU should decide how to access the N slots between two adjacent sensing events. If all N sub-problems \mathbf{P}_i achieve optimal simultaneously, then the original problem \mathbf{P} will be optimal.

B. Selective Access Strategy

We first analyze the property of interference caused by SU's transmission. Without loss of generality, we assume that SU senses channel *i* at time $t_0 = 0$. If the sensing result is "OFF" and SU decides to access this channel in the following *m*-th slot, then the expect interference to PU_i in the *m*-slot is

$$\phi_i(m) = \frac{1}{T_s} \int_{(m-1)T_s}^{mT_s} \frac{\lambda_i - \lambda_i e^{-(\lambda_i + \mu_i)\tau}}{\lambda_i + \mu_i} d\tau.$$
(8)

Similar to [7], we can obtain the following lemma.

Lemma 1: If the sensing result is "OFF", the interference caused by SU's transmission in the former slot is less than the one in the latter slot. That is, if n < m ($\forall n, m \in \mathbb{N}$), then $\phi_i(n) < \phi_i(m)$.

Since it can be easily obtained from (8), we omit the proof of this lemma. Based on lemma 1, we can obtain the following intuitive lemma directly.

Lemma 2: Once SU discovers spectrum hole, it should transmit consecutively in the following earlier slots (i.e., during $[0, \rho_i NT_s]$, where $\rho_i = 0, \frac{1}{N}, \frac{2}{N}, \dots, 1$).

Based on lemma 2, the SU knows how to access the channel qualitatively, but not quantitatively. In other words, the ratio ρ_i is unknown.

According to Lemma 2, SU's interference to PU_i is

$$I_i = k_i \cdot \frac{1}{T_P} \int_0^{\rho_i T_P} \frac{\lambda_i - \lambda_i e^{-(\lambda_i + \mu_i)\tau}}{\lambda_i + \mu_i} d\tau \tag{9}$$

where $k_i = \frac{\mu_i}{\mu_i + \lambda_i}$ is the probability of the sensing result being "OFF". Therefore, the sub-problem \mathbf{P}_i is equivalent to

$$\max_{\rho_i, T_P} \qquad U_i = k_i \rho_i \tag{10}$$

s.t.
$$\frac{k_i}{T_P} \int_0^{\rho_i T_P} \frac{\lambda_i - \lambda_i e^{-(\lambda_i + \mu_i)\tau}}{\lambda_i + \mu_i} d\tau \le C_i$$
 (11)

where $\rho_i = \{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\}$ and $T_P = NT_s > 0$.

It is very similar to our previous work [10], in which ρ_i is continuous variable. In [10], we have proved that:

- 1) If $T_P \leq T_c^i$ (the threshold T_c^i will be given latter), then $\rho_i = 1$ and SU's channel utilization is the maximal.
- 2) If $T_P > T_c^i$, then $\rho_i < 1$ and SU's channel utilization will decrease as T_P increases.

Remark: If T_P is small, during $[0, T_P]$, the probability p' that PU's idle state changes is very small. Thus, SU can access N slots (i.e., during $[0, T_P]$) and will not cause much interference to PU_i. As T_P increases, p' increases, especially at the end of duration $[0, T_P]$. Therefore, SU should reduce its transmission time and transmit as early as possible.

Furthermore, in [10], we have obtained that

$$T_{\rm c}^{i} = \frac{1}{\lambda_i + \mu_i} \left(\mathcal{W}\left(\frac{1}{m_i} e^{\frac{1}{m_i}}\right) - \frac{1}{m_i} \right)$$
(12)

where $m_i = \frac{C_i}{k_i(1-k_i)} - 1$ (when $C_i < k_i(1-k_i)$) and $\mathcal{W}(x)$ denotes the Lambert's W function [17]. Therefore, if λ_i and μ_i are big (i.e., channel's state changes fast) or C_i is small (i.e., interference constraint is strict), then T_c^i is small. It is in accord with intuition.

According to the above discussion, if $T_P \leq T_c^i$, SU's channel utilization on channel *i* is the maximal. Therefore, we have the following optimal access theorem.

Theorem 1: With PS strategy, if sensing period $T_s \leq \frac{T_c^*}{N}$, the optimal access strategy for SU to access channel *i* is that once SU discovers spectrum hole, it can greedily access all subsequent slots until channel *i* is probed next time. And then, SU's channel utilization on channel *i* is the maximal, which equals to channel *i*'s idle probability (i.e., $\frac{\mu_i}{\mu_i + \lambda_i}$).

The access strategy can be regarded as greedy access strategy. If $T_s \leq \min_{1 \leq i \leq N} \{T_c^i/N\}$, then the greedy access strategy can be adopted for all channels. We call it periodic sensing and selective access (PS-SA) strategy.

With PS-SA strategy, all sub-problems achieve optimal simultaneously. Therefore, SU's total channel utilization is maximal. However, since all channels are treated equally, most sensing opportunities are wasted on those channels that don't need to be sensed yet. Thus, the PS strategy is not efficient. Therefore, a selective sensing strategy, which makes SU first sense the channel that needs to be probed the most, is required.

IV. SELECTIVE SENSING AND SELECTIVE ACCESS

A. Selective Sensing Strategy

Based on the former discussion, we find that T_c^i , which is related to the channel's characteristic parameters (μ_i, λ_i) and interference constraint threshold (C_i) , reflects the frequency that channel *i* should be probed. Thus naturally, we propose a selective sensing strategy, which makes each channel *i* almost be probed periodically with period T_c^i . Particularly, SU senses the channel, whose "age" of last sensing result is closest to T_c^i . Mathematically, this selective sensing strategy leads to

$$CH = \arg\min_{1 \le i \le N} \left\{ p \times T_c^i - a_i T_s \right\}$$
(13)

where $a_i \in \mathbb{N}$ is the "age" (in terms of number of slots) of last sensing result of channel *i* and $p \in (0, 1)$ is a constant coefficient. Since when the sensing time interval is greater than T_c^i , SU's channel utilization will degrade. Thus, the parameter *p* is introduced in order to make SU sense each channel in advance. According to simulation result, we obtain that when p = 0.9, sensing period is the maximal for most situations. Thus, we choose p = 0.9.

The SS strategy is not strict periodic generally. However, since each channel will be sensed when sensing time interval is close to pT_c^i , the SS strategy for each channel can be regarded as periodic approximately.

B. Selective Access Strategy

According to lemma 2, once the spectrum opportunity is found, SU should access the channel as early as possible. With the SS strategy, if the sensing time interval for any channel *i* less than T_c^i , then the greedy access strategy is also suitable for the SS strategy. Since the SS strategy can be regarded as periodic approximately for each channel *i*, with the greedy access strategy, SU's channel utilization on channel *i* is about k_i , which equals to the one with PS-SA strategy. Therefore, similar to PS-SA strategy, we also adopt the simple greedy access strategy. On the other, since the channel with small T_c^i will be probed frequently (namely, fewer slots), the sensing period T_s could be larger than PS strategy. Therefore, with SS-SA strategy, SU could achieve the same channel utilization as the case with PS-SA strategy and meanwhile consume less time and energy to sense the channels.

It is noteworthy that unlike the PS-SA strategy, we could not give the expression of T_s . However, the approximate T_s can

be obtain by simulation. Given channels' parameters (μ_i, λ_i) , we can generate all channels' states and simulate the SS-SA strategy for different T_s . Then, we can obtain SU's channel utilization and its interference to each PU. The approximate T_s is the maximal T_s that makes the interference to all PUs not exceed their thresholds (C_i) .

V. SIMULATION RESULTS

A. Intuitive Sensing and Selective Access Strategy

We first introduce a reference strategy, that is, Intuitive Sensing and Selective Access (IS-SA) strategy. We consider an intuitive sensing strategy: SU first senses the channel whose state (ON/OFF) is most likely to change. Particularly, we assume that channel *i* was last sensed in slot t_i , then in slot $t(>t_i)$, the age of last sensing result is $a_i = t - t_i$. Thus, channel *i*'s state varying during the period of $((t_i - 1)T_s, (t-1)T_s)$ is equivalent to the holding time being less than a_iT_s . Since the holding times are exponentially distributed, the probability P_i that holding time being less than a_iT_s is

$$P_i = \int_0^{a_i T_s} \theta_i e^{-\theta_i t} dt = 1 - e^{-\theta_i a_i T_s} \tag{14}$$

where

$$\theta_i = \begin{cases} \mu_i, & \text{the last sensing result is "ON"} \\ \lambda_i, & \text{the last sensing result is "OFF"} \end{cases}$$
(15)

Thus, we can obtain the intuitive sensing strategy:

$$\max_{1 \le i \le N} \{P_i\} \Longleftrightarrow \max_{1 \le i \le N} \{a_i \theta_i\}.$$
 (16)

If the "age" of sensing result (i.e., a_i) is large or the channel states vary fast (i.e., θ_i is larger), the channel will be probed first. This is the same as intuition. However, it is apparent that IS strategy is invalid for different interference thresholds.

Similar to PS and SS strategies, SU can also adopt greedy access strategy if sensing period is suitable.

B. Example 1: Performance for Different Holding Times

We study the case that holding times are different. Particularly, we assume N = 5, $\lambda^{-1} = \mu^{-1} = [2, 4, 6, 8, 10]$ (second) and $C_i = 5\%$ ($\forall i$). Thus, we have $T_c = [0.464, 0.928, 1.392, 1.857, 2.321]$ (second).

The channel utilization for PS-SA, SS-SA and IS-SA strategies are shown in Fig. 2. Fig. 2 shows that SU's channel utilization on each channel is 50%, which equals to the idle probability, and SU's total channel utilization is the same for different strategies.

Fig. 3, Fig. 4 and Fig. 5 show the interference with PS-SA, SS-SA and IS-SA strategy, respectively. Fig. 3 shows when $T_s \leq 93$ (ms), the interference to each PU is less than the threshold (5%). Thus, the maximal sensing period is about 93 (ms), which is in accord with the theoretical value $\min \{T_c^i/N\} = 92.8$ (ms). Similarly, the maximal sensing period for SS-SA and IS-SA strategies are 184 and 183.5 (ms), which are approximately the same in this case. Therefore, SU consumes less time and energy to probe the channels by adopting SS-SA or IS-SA strategy.



Fig. 2. The channel utilization under PS-SA, SS-SA and IS-SA strategy.



Fig. 3. The interference under PS-SA strategy for different holding times.



Fig. 4. The interference under SS-SA strategy for different holding times.

C. Example 2: Performance for Different Thresholds C_i

We focus on the case that N = 5, $\lambda_i^{-1} = \mu_i^{-1} = 3$ ($\forall i$) and $C_i = [2\%, 4\%, 6\%, 8\%, 10\%]$. Therefore, $T_c = [254, 539, 865, 1242, 1689]$ (ms). Similar to Example 1, since the idle probability of each channel is 50%, SU's total channel utilization for each strategy is 2.5.

Since the parameters λ_i and μ_i are the same, with IS-SA strategy, all channels will be regarded as the same. Thus, IS-



Fig. 5. The interference under IS-SA strategy for different holding times.



Fig. 6. The interference under IS-SA (PS-SA) strategy for different C.



Fig. 7. The interference under SS-SA strategy for different C.

SA strategy is the same as PS-SA strategy and the five curves in Fig. 6 overlap each other. Due to $\min\{C_i\} = 2\%$, the maximal sensing period is about 51 (ms), which is in accord with the theoretical value. Since SS-SA strategy takes into account both channel's parameters and interference tolerances, these channels will not be regarded as the same any more. The interference with SS-SA strategy is illustrated in Fig. 7, which shows that the maximal sensing period is about 108 (ms), which is twice as much as IS-SA strategy. Thus in this case, SS-SA strategy is better than IS-SA and PS-SA strategies.

VI. CONCLUSION

In this paper, we propose a selective sensing and selective access (SS-SA) strategy for one slotted SU overlaying a nontime-slotted ON/OFF CTMC modeled multi-channel primary network. With SS strategy, each channel is probed approximate periodically with different periods according to the parameter T_c , which reflects the maximal period that each channel should be detected. If sensing period is suitable, SA strategy can be regarded as greedy access strategy. We also give two reference strategies (namely, PS-SA and IS-SA). Numerical simulations illustrate that T_c is a valid measurement to indicate how often each channel should be sensed, and with SS-SA strategy, SU can effectively utilize each channel and consume less energy and time for sensing than PS-SA and IS-SA strategies.

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