NAME: [Solutions]

CSCE 235 Quiz 1 (Solutions)
February 2, 2007

Determine whether each of the following proofs is a direct proof, a proof by contraposition, or a proof by contradiction.

1. **Proposition.** If n is an integer and $n^3 + 5$ is odd, then n is even.

   Proof. Suppose that n is odd. Then there exists an integer k such that $n = 2k + 1$. So

   $$n^3 + 5 = (2k + 1)^3 + 5 = (8k^3 + 12k^2 + 6k + 1) + 5 = 8k^3 + 12k^2 + 6k + 6 = 2(4k^3 + 6k^2 + 3k + 3).$$

   Taking $\ell = 4k^3 + 6k^2 + 3k + 3$, which is an integer, we see that $n^3 + 5 = 2\ell$, so $n^3 + 5$ is even. Therefore, if $n^3 + 5$ is odd, then n is even. □

   Solution. Let p be the statement "$n^3 + 5$ is odd" and let q be the statement "n is even". We are asked to prove that $p \to q$ is true for all integers n. In this proof, we first suppose that n is odd (that is, we suppose $\neg q$ is true), and then we reason by rules of inference to conclude that $n^3 + 5$ must be even (that is, $\neg p$ must be true). So we have proved $\neg q \to \neg p$, which is the *contrapositive* of the statement we are asked to prove. Therefore, this is a **proof by contraposition**.

2. **Proposition.** There is no smallest positive rational number.

   Proof. Assume that there is a smallest positive rational number; call it r. Since r is rational, we may write it as $r = p/q$ for some integers p and q. Now consider the number $r/2 = p/(2q)$. Since r is positive, $r/2$ is positive, and $r/2$ is smaller than r. Moreover, since q is an integer, $2q$ is an integer. Thus $r/2$ is rational. This contradicts our assumption that r is the smallest positive rational number; therefore, we conclude that there is no smallest positive rational number. □

   Solution. Here we are not asked to prove a statement of the form $p \to q$; rather, we are asked to prove that some object does not exist, so this is a "nonexistence" proof. Since there is no implication ($p \to q$) in the statement of the proposition we are asked to prove, it does not make sense to use a proof by contraposition, because contrapositives exist only for implications.

   In this proof we begin by assuming that the proposition is false. We then use logical reasoning to produce a *contradiction*: namely, if there is a smallest positive rational number r, then there is a number smaller than r which is both positive and rational, so r is not the smallest positive rational number. Since this proof supposes the proposition is false and then reaches a contradiction, it is a **proof by contradiction**.

3. **Proposition.** Every odd integer is the difference of two perfect squares. [Recall that a *perfect square*, commonly called just a *square*, is an integer m with the property that there exists an integer k such that $m = k^2$.]

   Proof. Let x be an odd integer. Then $x = 2k + 1$ for some integer k. So

   $$(k+1)^2 - k^2 = (k^2 + 2k + 1) - k^2 = 2k + 1 = x.$$

   Hence x is the difference of two perfect squares, namely, $(k+1)^2$ and k^2. $\square$

   Solution. This proposition is equivalent to the statement "If x is an odd integer, then x is the difference of two perfect squares." Letting p be the statement "x is an odd integer" and q be the statement "x is the difference of two perfect squares", we see that we are trying to prove the statement $p \to q$. In the proof, we begin by supposing that x is an odd integer (that is, we suppose that p is true), and through logical reasoning we arrive at the conclusion that x must be the difference of two perfect squares (that is, q must be true). Since we began by assuming the premises of the proposition are true and found that this implied the conclusion of the proposition is true, this is a **direct proof**.

4. **Proposition.** If ten balls are chosen from a box containing only red balls, yellow balls, and green balls, then at least four of the chosen balls are of the same color.

   Proof. Suppose that ten balls are chosen from the box and that no four of the chosen balls are of the same color. Then the chosen balls must include no more than three balls of any one color. Since there are only three colors, the number of chosen balls can be no greater than $3 \times 3 = 9$. But ten balls were chosen. Hence at least four of the chosen balls are of the same color. $\square$

   Solution. Let p be the statement "ten balls are chosen from the box" and q be the statement "at least four of the chosen balls are of the same color". We are asked to prove $p \to q$.

   Notice that in this proof we begin by supposing that ten balls are chosen from the box (that is, that p is true) and that no four of the chosen balls are of the same color (that is, that $\neg q$ is true). We then reason logically to conclude that at most nine balls were chosen from the box (so that $\neg p$ is true). This means that we have $p \wedge \neg p$, which is a *contradiction*. In other words, we began by supposing that the premises of the proposition are true and that the conclusion is false, and we arrived at a contradiction. So this is a **proof by contradiction**. This proof shows that $(p \wedge \neg q) \to (p \wedge \neg p)$.

   This proof could very easily be rewritten to be a proof by contraposition. If we remove the assumption at the beginning of the proof that ten balls are chosen from the box, so that our only assumption is that no four of the chosen balls are of the same color, we can still logically deduce that no more than nine balls were chosen from the box. In this case, the proof shows that $\neg q \to \neg p$, which is the contrapositive of the original statement. Many proofs by contradiction can be slightly simplified in a similar way (by removing unnecessary assumptions) to produce a proof by contraposition. Often the resulting proof is considered a simpler and more elegant proof than the original proof by contradiction, but this is just a matter of mathematical taste and does not affect the validity of the proof.