CSCE 235: Introduction to Discrete Structures
Prerequisite test (solutions)
January 12, 2007

Name: [SOLUTIONS]

Have you taken CSCE 155/155H? If so, what was your grade?

Have you taken MATH 106/108H?

This test consists of five questions worth a total of 30 points. You have 20 minutes to complete the test. You must show all steps, including any computations or explanations, that led you to your answers.

1. (5 points) Let $\max(a, b)$ be the function that returns the maximum of the two numbers a and b, and let $\min(a, b)$ be the function that returns the minimum of the two numbers a and b. What is the value of

$$\max\Big(a, \min\Big(\max\big(b, \min(a, b)\big), \min\big(a, \max(a, b)\big)\Big)\Big)?$$

Solution. If $a \leq b$, then $\min(a, b) = a$, so

$$\max\big(b, \min(a, b)\big) = \max(b, a) = b.$$

On the other hand, if $a > b$, then $\min(a, b) = b$, so

$$\max\big(b, \min(a, b)\big) = \max(b, b) = b.$$

In either case, then, we see that

$$\max\big(b, \min(a, b)\big) = b.$$

Similar reasoning shows that

$$\min\big(a, \max(a, b)\big) = a$$

and

$$\max\big(a, \min(b, a)\big) = a.$$

Therefore,

$$\max\Big(a, \min\Big(\underbrace{\max\big(b, \min(a, b)\big)}_{b}, \underbrace{\min\big(a, \max(a, b)\big)}_{a}\Big)\Big) = \max\big(a, \min(b, a)\big) = a.$$

2. (4 points) In *The Hitchhiker's Guide to the Galaxy*, Douglas Adams says that the answer to the Question of Life, the Universe, and Everything is 42. Express the decimal number 42 in

(a) binary.

Solution. The decimal system is based on powers of 10, so the decimal number 42 means $(4 \times 10^1) + (2 \times 10^0)$. The binary system works the same way, but it is based on powers of 2. Just as the digits in the decimal system can range from 0 to 9 (one less than 10), the digits in the binary system can range from 0 to 1 (one less than 2).

We want to express 42 as a sum of powers of 2. We can do this as follows:

2^5	2^4	2^3	2^2	2^1	2^0
32	16	8	4	2	1
1	0	1	0	1	0

since

$$(1 \times 2^5) + (0 \times 2^4) + (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (0 \times 2^0) = 32 + 8 + 2 = 42.$$

Therefore, the decimal number 42 is expressed in binary as 101010.

(b) base 5.

Solution. In base 5, we express a number as a sum of powers of 5, with digits ranging from 0 to 4 (one less than 5). To express the decimal number 42 in base 5, we find

5^2	5^1	5^0
25	5	1
1	3	2

since

$$(1 \times 5^2) + (3 \times 5^1) + (2 \times 5^0) = 25 + 15 + 2 = 42.$$

Therefore, the decimal number 42 is expressed in base 5 as 132.

3. (4 points)

(a) A document is enlarged by a factor of 150% on a photocopier. By what factor must the copy be reduced to produce a copy the same size as the original?

Solution. Let x represent the original size of the document. The copy was made by setting the photocopier to 150%, so the size of the copy is $1.5x$. We want to find the factor y at which we should set the photocopier so that the copy of the copy will be the same size as the original document; in other words, we want to find y such that

$$(1.5x)y = x.$$

We may make the reasonable assumption that $x \neq 0$, so we can divide both sides of this equation by x to obtain

$$1.5y = 1.$$

Solving for y, we find that $y = 0.666\ldots = 2/3$, so the factor by which we must reduce the copy is $66\frac{2}{3}\%$.

(b) Sam and Kim both started new jobs at the beginning of 2005, with the same starting salary. At the end of 2005, Sam received a 10% raise, while Kim's salary was reduced by 5%. At the end of 2006, however, it was Kim who received the 10% raise, and Sam's salary fell by 5%. After these two sets of salary adjustments, whose salary is larger?

Solution. Let x represent the salary at the beginning of 2005, so both Sam and Kim had a salary of x at the beginning of 2005. At the end of 2005, Sam's salary increases by 10%, so it becomes 110% of its original value; therefore, Sam's salary at the beginning of 2006 is $1.10x$. Similarly, Kim's salary drops by 5% at the end of 2005, so it becomes 95% of its original value; therefore, Kim's salary at the beginning of 2006 is $0.95x$.

Now, at the end of 2006, Sam's salary is decreased by 5%, so its value at the beginning of 2007 is 95% of its value at the beginning of 2006. Thus, at the beginning of 2007, Sam's salary is $0.95(1.10x)$. On the other hand, Kim's salary is raised by 10% at the end of 2006, so its value at the beginning of 2007 is 110% of its value at the beginning of 2006; hence Kim's salary at the beginning of 2007 is $1.10(0.95x)$.

We are asked to compare the two salaries after these two sets of salary adjustments, that is, at the beginning of 2007. At this time, Sam's salary is $0.95(1.10x)$ and Kim's salary is $1.10(0.95x)$. Since multiplication is commutative, these two quantities are equal. Therefore, Sam and Kim have the same salary after these two sets of salary adjustments.

4. (8 points) What is the output of the following pseudocode?

```
n := 13
while n > 1 do
    output n
    if n is even then
        n := n/2
    else
        n := 3n + 1
    end
end
```

Solution. The first line initializes the variable n to the value 13. The program then enters a loop, in which it will remain as long as the value of n is greater than 1. This condition is initially satisfied, since $13 > 1$, so program begins executing the statements inside the loop.

The first statement in the loop causes the program to output the value of n, which at this point is 13. Next, a conditional statement is reached, which tests whether the value of n is even. Currently the value of n is 13, which is odd, so the statement following the keyword **else** is executed, which computes the value $3n + 1 = 3 \times 13 + 1 = 40$ and stores this value in the variable n.

Now control jumps back to the loop condition again, and the program tests whether the value of n is greater than 1. It is, since $40 > 1$, so the program executes the statements inside the loop again. First, the value of n is output; currently this value is 40. Then the evenness of n is tested. This condition is now true, so the program computes the value $n/2 = 40/2 = 20$ and stores this value in the variable n. Again, control jumps back to the loop condition.

This process continues for as long as the value in n is greater than 1. Therefore, we see that the output of the program will be the list of numbers

$$13,\ 40,\ 20,\ 10,\ 5,\ 16,\ 8,\ 4,\ 2.$$

It is important to note that the value 1 will not be output, because when the value of n is 1, the loop condition is false (as $1 \not> 1$), and so the statements inside the loop will not be executed.

5. (9 points) An economist predicts, "If the United States does not reduce its trade deficit, then the country will fall into a recession."

   (a) Based on this prediction, is it reasonable to conclude that "if the United States falls into a recession, it must be because the trade deficit was not reduced"? Why or why not?

   Solution. No. The country may fall into a recession for some other reason. This statement is the *converse* of the original prediction.

   (b) Based on this prediction, is it reasonable to conclude that "if the United States does not fall into a recession, then the trade deficit must have been reduced"? Why or why not?

   Solution. Yes. The only way the United States can avoid falling into a recession is by reducing its trade deficit. (Reducing the trade deficit is not sufficient to guarantee that the country will not experience a recession, but it is necessary.) If the trade deficit had not been reduced, then according to the economist's prediction the United States would have fallen into a recession. The statement in part (b) is the *contrapositive* of the original prediction.

   (c) Based on this prediction, is it reasonable to conclude that "if the United States reduces its trade deficit, then the country will not fall into a recession"? Why or why not?

   Solution. No. Some other crisis may plunge the country into a recession. This statement is the *inverse* of the original prediction, and as such it is logically equivalent to the statement in part (a).