CSCE 235: Introduction to Discrete Structures
Homework assignment 6 (solutions)
Assigned Monday, April 2, 2007
Due Monday, April 9, 2007

Problem 1. (12 points) Let $a_n = 2^n + 5 \cdot 3^n$ for $n = 0, 1, 2, \ldots$.

(a) Find a_0, a_1, a_2, a_3, and a_4.
(b) Show that $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$, and $a_4 = 5a_3 - 6a_2$.
(c) Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers n with $n \geq 2$.

Solution.

(a)
$$\begin{aligned} a_0 &= 2^0 + 5 \cdot 3^0 = 6, \\ a_1 &= 2^1 + 5 \cdot 3^1 = 17, \\ a_2 &= 2^2 + 5 \cdot 3^2 = 49, \\ a_3 &= 2^3 + 5 \cdot 3^3 = 143, \\ a_4 &= 2^4 + 5 \cdot 3^4 = 421. \end{aligned}$$

(b)
$$\begin{aligned} a_2 &= 49 = 5 \cdot 17 - 6 \cdot 6 = 5a_1 - 6a_0, \\ a_3 &= 143 = 5 \cdot 49 - 6 \cdot 17 = 5a_2 - 6a_1, \\ a_4 &= 421 = 5 \cdot 143 - 6 \cdot 49 = 5a_3 - 6a_2. \end{aligned}$$

(c) For $n \geq 2$,

$$\begin{aligned} a_n &= 2^n + 5 \cdot 3^n \\ &= 2 \cdot 2^{n-1} + 5 \cdot 3 \cdot 3^{n-1} \\ &= (5-3) \cdot 2^{n-1} + (25-10) \cdot 3^{n-1} \\ &= 5 \cdot 2^{n-1} + 25 \cdot 3^{n-1} - 3 \cdot 2^{n-1} - 10 \cdot 3^{n-1} \\ &= 5 \cdot 2^{n-1} + 25 \cdot 3^{n-1} - 3 \cdot 2 \cdot 2^{n-2} - 10 \cdot 3 \cdot 3^{n-2} \\ &= 5 \cdot 2^{n-1} + 25 \cdot 3^{n-1} - 6 \cdot 2^{n-2} - 30 \cdot 3^{n-2} \\ &= 5(2^{n-1} + 5 \cdot 3^{n-1}) - 6(2^{n-2} + 5 \cdot 3^{n-2}) \\ &= 5a_{n-1} - 6a_{n-2}. \end{aligned}$$

Problem 2. (16 points) Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ if

(a) $a_n = 0$?
(b) $a_n = 1$?
(c) $a_n = 2^n$?
(d) $a_n = 4^n$?
(e) $a_n = n4^n$?
(f) $a_n = 2 \cdot 4^n + 3n4^n$?
(g) $a_n = (-4)^n$?
(h) $a_n = n^2 4^n$?

Solution.

(a) Yes; $a_n = 0 = 8 \cdot 0 - 16 \cdot 0 = 8a_{n-1} - 16a_{n-2}$.

(b) No; $a_n = 1 \neq 8 \cdot 1 - 16 \cdot 1 = 8a_{n-1} - 16a_{n-2}$.

(c) No; $a_n = 2^n \neq 0 = 8 \cdot 2^{n-1} - 16 \cdot 2^{n-2} = 8a_{n-1} - 16a_{n-2}$.

(d) Yes; $a_n = 4^n = 2 \cdot 4^n - 4^n = 8 \cdot 4^{n-1} - 16 \cdot 4^{n-2} = 8a_{n-1} - 16a_{n-2}$.

(e) Yes;

$$\begin{aligned} a_n &= n \cdot 4^n \\ &= n(2 \cdot 4^n - 4^n) - 2 \cdot 4^n + 2 \cdot 4^n \\ &= 2 \cdot 4^n(n-1) - 4^n(n-2) \\ &= 8(n-1) \cdot 4^{n-1} - 16(n-2) \cdot 4^{n-2} \\ &= 8a_{n-1} - 16a_{n-2}. \end{aligned}$$

(f) Yes;

$$\begin{aligned} a_n &= 2 \cdot 4^n + 3n \cdot 4^n \\ &= 4^n(2+3n) \\ &= 4^n\left[4 + 6(n-1) - 2 - 3(n-2)\right] \\ &= 4 \cdot 4^n + 6(n-1) \cdot 4^n - 2 \cdot 4^n - 3(n-2) \cdot 4^n \\ &= 8\left[2 \cdot 4^{n-1} + 3(n-1) \cdot 4^{n-1}\right] - 16\left[2 \cdot 4^{n-2} + 3(n-2) \cdot 4^{n-2}\right] \\ &= 8a_{n-1} - 16a_{n-2}. \end{aligned}$$

(g) No; for example, $a_2 = (-4)^2 = 16 \neq -48 = 8(-4) - 16 \cdot 1 = 8a_1 - 16a_0$.

(h) No; for example, $a_2 = 2^2 \cdot 4^2 = 64 \neq 32 = 8 \cdot 4 - 16 \cdot 0 = 8a_1 - 16a_0$.

Problem 3. (12 points) Assume that the population of the world in 2007 is 6.6 billion and that the population will grow at the constant rate of 1.14% each year.

(a) Set up a recurrence relation for the population of the world n years after 2007.
(b) Find an explicit formula for the population of the world n years after 2007.
(c) What will the population of the world be in 2027?
(d) When will the world population reach 10 billion?

Solution.

(a) Let P_n represent the world population n years after 2007. Then we have, for $n = 1, 2, 3, \ldots,$

$$P_n = 1.0114P_{n-1}.$$

(b)
$$P_n = (6.6 \times 10^9)(1.0114)^n.$$

(c)
$$P_{20} = (6.6 \times 10^9)(1.0114)^{20} \approx 8.28 \text{ billion}.$$

(d) We find that

$$(6.6 \times 10^9)(1.0114)^{36.656\ldots} \approx 10 \text{ billion},$$

so the world population will reach 10 billion about 36.7 years after 2007. How should this be interpreted? One possibility is to read "after 2007" as "after January 1, 2007," in which case our answer is August 27, 2043. Another way is to reason that the population in 2043 is just under 10 billion, and the population in 2044 is just over 10 billion, so the actual date must be somewhere between the "census dates" of 2043 and 2044.

Problem 4. (12 points) A vending machine dispensing books of stamps accepts only dollar coins, \$1 bills, and \$5 bills.

(a) Find a recurrence relation for the number of ways to deposit n dollars into the machine, where the order in which the coins and bills are deposited matters.
(b) What are the initial conditions?
(c) How many ways are there to deposit \$10 for a book of stamps?

Solution.

(a) Let A_n represent the number of ways to deposit n dollars into the machine.

Suppose we wish to deposit n dollars into the machine, where $n \geq 5$. The first piece of currency we deposit must be either a dollar coin, a \$1 bill, or a \$5 bill. If we begin by depositing a dollar coin, then we have A_{n-1} ways to deposit the rest of the money. If instead we first deposit a \$1 bill, then we also have A_{n-1} ways to deposit the rest of the money. Finally, if we start by depositing a \$5 bill, then we have A_{n-5} ways to deposit the rest of the money.

In total, then, the number of ways to deposit n dollars into the machine is

$$A_n = A_{n-1} + A_{n-1} + A_{n-5} = 2A_{n-1} + A_{n-5},$$

for $n = 5, 6, 7, \ldots$.

(b) Our recurrence relation works only for $n \geq 5$, and requires a value for A_{n-5}, so our initial conditions are

$$A_0 = 1, \qquad A_1 = 2, \qquad A_2 = 4, \qquad A_3 = 8, \qquad A_4 = 16.$$

(c) We use our recurrence relation and initial conditions to compute

$$\begin{aligned}
A_5 &= 2A_4 + A_0 = 2 \cdot 16 + 1 = 33, \\
A_6 &= 2A_5 + A_1 = 2 \cdot 33 + 2 = 68, \\
A_7 &= 2A_6 + A_2 = 2 \cdot 68 + 4 = 140, \\
A_8 &= 2A_7 + A_3 = 2 \cdot 140 + 8 = 288, \\
A_9 &= 2A_8 + A_4 = 2 \cdot 288 + 16 = 592, \\
A_{10} &= 2A_9 + A_5 = 2 \cdot 592 + 33 = 1217.
\end{aligned}$$

So there are 1217 ways to deposit \$10 into the machine.

Problem 5. (10 points) Find the solution to the recurrence relation

$$a_n = 7a_{n-1} - 10a_{n-2} \qquad \text{for } n \geq 2,$$

with initial conditions $a_0 = 2$ and $a_1 = 1$.

Solution. The characteristic polynomial of this recurrence relation is $r^2 - 7r + 10$, which we notice factors as $(r-5)(r-2)$. Therefore, the roots of this polynomial, i.e., the characteristic roots of the recurrence relation, are $r = 5$ and $r = 2$. So the general solution to the recurrence relation is

$$a_n = \alpha_1 \cdot 5^n + \alpha_2 \cdot 2^n$$

for some constants α_1 and α_2. The initial conditions give us

$$\begin{aligned} a_0 &= 2 = \alpha_1 \cdot 5^0 + \alpha_2 \cdot 2^0 = \alpha_1 + \alpha_2, \\ a_1 &= 1 = \alpha_1 \cdot 5^1 + \alpha_2 \cdot 2^1 = 5\alpha_1 + 2\alpha_2. \end{aligned}$$

Solving this system of linear equations yields

$$\alpha_1 = -1, \qquad \alpha_2 = 3.$$

So the solution to this recurrence relation with these initial conditions is

$$a_n = -5^n + 3 \cdot 2^n.$$

Problem 6. (10 points) Find the solution to the recurrence relation

$$a_n = -6a_{n-1} - 9a_{n-2} \qquad \text{for } n \geq 2,$$

with initial conditions $a_0 = 3$ and $a_1 = -3$.

Solution. The characteristic polynomial of this recurrence relation is $r^2 + 6r + 9$, which factors as $(r+3)(r+3)$, so the characteristic root is $r = -3$, with multiplicity 2. Thus the general solution to the recurrence relation is

$$a_n = \alpha_1 \cdot (-3)^n + \alpha_2 \cdot n \cdot (-3)^n$$

for some constants α_1 and α_2. The initial conditions give us

$$\begin{aligned} a_0 &= 3 = \alpha_1(-3)^0 + \alpha_2 \cdot 0 \cdot (-3)^0 = \alpha_1, \\ a_1 &= -3 = \alpha_1(-3)^1 + \alpha_2 \cdot 1 \cdot (-3)^1 = -3\alpha_1 - 3\alpha_2, \end{aligned}$$

so $\alpha_1 = 3$ and $\alpha_2 = -2$. Therefore the solution to this recurrence relation under the given initial conditions is

$$a_n = 3(-3)^n - 2n(-3)^n.$$

Problem 7. (10 points) Find the solution to the recurrence relation

$$a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3} \qquad \text{for } n \geq 3,$$

with initial conditions $a_0 = 7$, $a_1 = -4$, and $a_2 = 8$.

Hint: You will need to use Theorem 3 on page 465 of the textbook.

Solution. The characteristic polynomial is $r^3 - 2r^2 - 5r + 6$, which factors as $(r-3)(r-1)(r+2)$. Hence the characteristic roots are $r = 3$, $r = 1$, and $r = -2$, and the general solution to the recurrence relation is

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 1^n + \alpha_3 \cdot (-2)^n$$

for some constants α_1, α_2, and α_3. The initial conditions give us

$$\begin{aligned} a_0 &= 7 = \alpha_1 \cdot 3^0 + \alpha_2 + \alpha_3 \cdot (-2)^0 = \alpha_1 + \alpha_2 + \alpha_3, \\ a_1 &= -4 = \alpha_1 \cdot 3^1 + \alpha_2 + \alpha_3 \cdot (-2)^1 = 3\alpha_1 + \alpha_2 - 2\alpha_3, \\ a_2 &= 8 = \alpha_1 \cdot 3^2 + \alpha_2 + \alpha_3 \cdot (-2)^2 = 9\alpha_1 + \alpha_2 + 4\alpha_3. \end{aligned}$$

Solving this system of linear equations yields

$$\alpha_1 = -1, \qquad \alpha_2 = 5, \qquad \alpha_3 = 3.$$

So the solution to this recurrence relation with the given initial conditions is

$$a_n = -3^n + 5 + 3 \cdot (-2)^n.$$

Problem 8. (10 points) The Lucas numbers satisfy the recurrence relation

$$L_n = L_{n-1} + L_{n-2} \qquad \text{for } n \geq 2,$$

with the initial conditions $L_0 = 2$ and $L_1 = 1$. Find an explicit formula for the Lucas numbers.

Solution. The Lucas numbers satisfy the same recurrence relation as the Fibonacci numbers. In class we showed that the general solution to this recurrence relation is

$$L_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constants α_1 and α_2. Using the initial conditions given, we get

$$\begin{aligned} L_0 &= 2 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^0 = \alpha_1 + \alpha_2, \\ L_1 &= 1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^1 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^1 = \left(\frac{1+\sqrt{5}}{2}\right)\alpha_1 + \left(\frac{1-\sqrt{5}}{2}\right)\alpha_2. \end{aligned}$$

Solving this system of equations, we obtain

$$\alpha_1 = 1 \qquad \text{and} \qquad \alpha_2 = 1.$$

So an explicit formula for the nth Lucas number is

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$