CSCE 235: Introduction to Discrete Structures
Homework assignment 4 (solutions)
Assigned Wednesday, February 21, 2007
Due Wednesday, February 28, 2007

Problem 1. (16 points) For each of the following functions, give the domain, codomain, and range of the function; specify whether or not it is one-to-one, whether or not it is onto, whether or not it is bijective, and whether or not it is invertible; and if it is invertible, give its inverse.

(a) $f : \mathbf{Q} \to \mathbf{Q}$ given by $f(x) = \frac{1}{2}x^2 - 4$.

(b) $g : \mathbf{R} \to \mathbf{R}$ given by $g(x) = x^5$.

(c) $h : \mathbf{R} \to \mathbf{Z}$ given by $h(x) = \lfloor x \rfloor$.

(d) $\phi : \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ given by $\phi(n) = (n, n)$.

Solution.

(a) The domain of f is $\mathbf{Q}$; its codomain is $\mathbf{Q}$; and its range is the set

$$\left\{ \frac{a^2}{2b^2} - 4 \;\middle|\; a \in \mathbf{Z} \wedge b \in \mathbf{Z} \right\}.$$

(It is incorrect to say that the range of f is $\{ y \in \mathbf{Q} \mid y \geq -4 \}$, since, for example, -3 is not in the range of f.) This function is not one-to-one; for example, $f(-1) = f(1)$. This function is not onto; for example, no rational value of x satisfies $f(x) = -5$. Since it is not one-to-one and not onto, it is not bijective, which means it is not invertible.

(b) The domain of g is $\mathbf{R}$; its codomain is $\mathbf{R}$; and its range is also $\mathbf{R}$. This function is one-to-one, since if $x \neq y$ we have $x^5 \neq y^5$, that is, $g(x) \neq g(y)$, which means that no two distinct real numbers will be mapped by g to the same number. This function is onto, since for any given real number x we have $g(\sqrt[5]{x}) = x$. Since this function is both one-to-one and onto, it is bijective, which means that it is invertible. The inverse of the function g is given by $g^{-1}(x) = \sqrt[5]{x}$.

(c) The domain of h is $\mathbf{R}$; its codomain is $\mathbf{Z}$; and its range is also $\mathbf{Z}$. This function is not one-to-one, since for example $h(\pi) = 3$ and $h(\sqrt{10}) = 3$, but $\pi \neq \sqrt{10}$. This function is onto, since for any integer n we have $h(n) = n$. Since this function is not one-to-one, it is not bijective, which means it is not invertible.

(d) The domain of ϕ is $\mathbf{N}$; its codomain is $\mathbf{N} \times \mathbf{N}$; and its range is the set

$$\{ (a, b) \in \mathbf{N} \times \mathbf{N} \mid a = b \}.$$

This function is one-to-one, since no two distinct natural numbers will be mapped by ϕ to the same ordered pair. It is not onto, since, for example, no natural number n satisfies $\phi(n) = (1, 2)$. Since it is not onto, it is not bijective, which means it is not invertible.

Problem 2. (8 points) Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x^2 + 1$ and let $g : \mathbf{R} \to \mathbf{R}$ be given by $g(x) = 3x + 2$.

(a) Find $f \circ g$.
(b) What is the value of $(f \circ g)(1)$? of $(f \circ g)(-1)$?
(c) Find $g \circ f$.
(d) What is the value of $(g \circ f)(1)$? of $(g \circ f)(-1)$?

Solution.

(a) $f \circ g = (3x + 2)^2 + 1 = 9x^2 + 12x + 5$.

(b)
$$(f \circ g)(1) = 9 \cdot 1^2 + 12 \cdot 1 + 5 = 26;$$
$$(f \circ g)(-1) = 9(-1)^2 + 12(-1) + 5 = 2.$$

(c) $g \circ f = 3(x^2 + 1) + 2 = 3x^2 + 5$.

(d)
$$(g \circ f)(1) = 3 \cdot 1^2 + 5 = 8;$$
$$(g \circ f)(-1) = 3(-1)^2 + 5 = 8.$$

Problem 3. (10 points) Find these values.

(a) $\lfloor 1.02 \rfloor$
(b) $\lceil 1.02 \rceil$
(c) $\lfloor -3.8 \rfloor$
(d) $\lceil -3.8 \rceil$
(e) $8!$
(f) $n!/(n-4)!$, where n is an integer greater than 4

Solution.

(a) $\lfloor 1.02 \rfloor = 1$.

(b) $\lceil 1.02 \rceil = 2$.

(c) $\lfloor -3.8 \rfloor = -4$.

(d) $\lceil -3.8 \rceil = -3$.

(e) $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40\,320$.

(f)
$$\begin{aligned}\frac{n!}{(n-4)!} &= \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)\cdot\dots\cdot 3 \cdot 2 \cdot 1}{(n-4)(n-5)\cdot\dots\cdot 3 \cdot 2 \cdot 1} \\ &= n(n-1)(n-2)(n-3) \\ &= n^4 - 6n^3 + 11n^2 - 6n.\end{aligned}$$

Problem 4. (20 points) Let $P(n)$ be the statement that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for the positive integer n. The parts of this problem outline a proof using mathematical induction to show that this formula is true for all positive integers n.

(a) What is the statement $P(1)$?
(b) Show that $P(1)$ is true, completing the basis step of the proof.
(c) What is the inductive hypothesis?
(d) What do you need to prove in the inductive step?
(e) Complete the inductive step.
(f) Explain why these steps show that this formula is true whenever n is a positive integer.

Solution.

(a) $P(1)$ is the statement that

$$1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}.$$

(b) $P(1)$ is true, because

$$1^2 = 1 = \frac{6}{6} = \frac{1 \cdot 2 \cdot 3}{6} = \frac{1(1+1)(2 \cdot 1 + 1)}{6}.$$

(c) The inductive hypothesis is the assumption that $P(k)$ is true for some positive integer k, i.e., that

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

(d) In the inductive step we need to prove that if the inductive hypothesis holds [i.e., if $P(k)$ is true], then $P(k+1)$ is true.

(e) When we add $(k+1)^2$ to both sides of the equation in part (c), we get

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{2k^3 + 3k^2 + k}{6} + \frac{6k^2 + 12k + 6}{6} \\
&= \frac{2k^3 + 9k^2 + 13k + 6}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}.
\end{aligned}$$

This shows that $P(k+1)$ is true under the assumption that $P(k)$ is true.

(f) We have completed the basis step and the inductive step. In the basis step, part (b), we showed that the formula is true if $n = 1$, and in the inductive step, part (e), we showed that if the formula is true for some positive integer k then it is also true for $k+1$. Therefore, by the principle of mathematical induction, the formula must be true for all positive integers.

□

Problem 5. (20 points) Prove that for every positive integer n,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Solution. Let $P(n)$ be the proposition that $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = n(n+1)(n+2)/3$ for the integer n.

Basis step. $P(1)$ is true because

$$1 \cdot 2 = 2 = \frac{6}{3} = \frac{1 \cdot 2 \cdot 3}{3} = \frac{1(1+1)(1+2)}{3}.$$

Inductive step. For the inductive hypothesis, we assume that $P(k)$ is true for some positive integer k, that is, we assume that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}. \tag{1}$$

To carry out the inductive step, we need to show that if $P(k)$ is true, then $P(k+1)$ is also true. That is, we must show that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (k+1)[(k+1)+1] = \frac{(k+1)[(k+1)+1][(k+1)+2]}{3},$$

i.e.,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}, \tag{2}$$

assuming the inductive hypothesis $P(k)$. If we add $(k+1)(k+2)$ to both sides of Equation (1), we get

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} \\ &= \frac{k[(k+1)(k+2)] + 3[(k+1)(k+2)]}{3} \\ &= \frac{(k+3)(k+1)(k+2)}{3}. \end{aligned}$$

This shows that Equation (2) is true if we assume the inductive hypothesis, which completes the inductive step.

We have completed both the basis step and the inductive step. That is, we have shown that $P(1)$ is true and the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k. Consequently, by the principle of mathematical induction we can conclude that $P(n)$ is true for all positive integers n, i.e., $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = n(n+1)(n+2)/3$ for all positive integers n. □

Problem 6. (20 points) Let $Q(n)$ be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps. The parts of this problem outline a strong induction proof that $Q(n)$ is true for $n \geq 8$.

(a) Show that the statements $Q(8)$, $Q(9)$, and $Q(10)$ are true, completing the basis step of the proof.
(b) What is the inductive hypothesis of the proof?
(c) What do you need to prove in the inductive step?
(d) Complete the inductive step.
(e) Explain why these steps show that this statement is true whenever $n \geq 8$.

Solution.

(a) $Q(8)$ is true, since a postage of 8 cents can be formed with one 3-cent stamp and one 5-cent stamp. $Q(9)$ is true, since a postage of 9 cents can be formed with three 3-cent stamps. $Q(10)$ is true, since a postage of 10 cents can be formed with two 5-cent stamps.

(b) The inductive hypothesis is the statement that $Q(j)$ is true for all integers j with $8 \leq j \leq k$, where k is an integer with $k \geq 10$. That is, we assume that we can form postage of j cents, where $8 \leq j \leq k$. (In other words, we are assuming that for some particular positive integer $k \geq 10$, we are able to form postage of any amount from 8 cents up to k cents.)

(c) To complete the inductive step, we need to show that under this assumption, $Q(k+1)$ is true, that is, we can form postage of $k+1$ cents.

(d) Using the inductive hypothesis, we can assume that $Q(k-2)$ is true because $k-2 \geq 8$, that is, we can form postage of $k-2$ cents using just 3-cent stamps and 5-cent stamps. To form postage of $k+1$ cents, we need only add another 3-cent stamp to the stamps we used to form postage of $k-2$ cents. Hence, we have shown that if the inductive hypothesis is true, then $Q(k+1)$ is also true.

(e) We showed directly that $Q(8)$, $Q(9)$, and $Q(10)$ are true, and in the inductive step we showed that if $Q(j)$ is true for all $8 \leq j \leq k$ with $k \geq 10$ then $Q(k+1)$ is true. By the principle of strong induction, then, $Q(n)$ is true for all integers $n \geq 8$. □

Problem 7. (8 points) Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$ if $f(n)$ is defined recursively by $f(0) = 1$ and for $n = 1, 2, 3, \ldots$,

(a) $f(n) = f(n-1) + 2$.
(b) $f(n) = \big(f(n-1) + 1\big)^2$.

Solution.

(a)
$$\begin{aligned} f(1) &= f(0) + 2 = 1 + 2 = 3, \\ f(2) &= f(1) + 2 = 3 + 2 = 5, \\ f(3) &= f(2) + 2 = 5 + 2 = 7, \\ f(4) &= f(3) + 2 = 7 + 2 = 9. \end{aligned}$$

(b)
$$\begin{aligned} f(1) &= \big(f(0) + 1\big)^2 = (1+1)^2 = 2^2 = 4, \\ f(2) &= \big(f(1) + 1\big)^2 = (4+1)^2 = 5^2 = 25, \\ f(3) &= \big(f(2) + 1\big)^2 = (25+1)^2 = 26^2 = 676, \\ f(4) &= \big(f(3) + 1\big)^2 = (676+1)^2 = 677^2 = 458\,329. \end{aligned}$$

Problem 8. (8 points) Find $f(2)$, $f(3)$, $f(4)$, and $f(5)$ if $f(n)$ is defined recursively by $f(0) = -1$, $f(1) = 2$, and for $n = 2, 3, 4, \ldots$,

(a) $f(n) = f(n-1) + 3f(n-2)$.
(b) $f(n) = f(n-2)/f(n-1)$.

Solution.

(a)

$$\begin{aligned} f(2) &= f(1) + 3f(0) = 2 + 3(-1) = -1, \\ f(3) &= f(2) + 3f(1) = -1 + 3(2) = 5, \\ f(4) &= f(3) + 3f(2) = 5 + 3(-1) = 2, \\ f(5) &= f(4) + 3f(3) = 2 + 3(5) = 17. \end{aligned}$$

(b)

$$\begin{aligned} f(2) &= f(0)/f(1) = (-1)/2 = -\tfrac{1}{2}, \\ f(3) &= f(1)/f(2) = 2/(-\tfrac{1}{2}) = -4, \\ f(4) &= f(2)/f(3) = (-\tfrac{1}{2})/(-4) = \tfrac{1}{8}, \\ f(5) &= f(3)/f(4) = (-4)/(\tfrac{1}{8}) = -32. \end{aligned}$$