CSCE 235: Introduction to Discrete Structures
**Homework assignment 3** (solutions)
Assigned Wednesday, February 7, 2007
Due Monday, February 12, 2007

**Problem 1.** (15 points) Prove that for any integer $n$ there does not exist an integer $k$ such that $n^2 = 3k + 2$.

**Solution.** We are trying to prove that no square leaves a remainder of 2 when divided by 3. If any integer is divided by 3, the remainder is either 0, 1, or 2. Therefore every integer $n$ can be written as $3r$, $3r + 1$, or $3r + 2$ for some integer $r$; moreover, $n$ can be written in exactly one of these three forms, depending on what remainder is left when $n$ is divided by 3. We shall consider these three cases separately.

**Case 1:** Suppose $n = 3r$ for some integer $r$. Then

$$n^2 = (3r)^2 = 9r^2 = 3(3r^2),$$

so $n^2 = 3s$, where $s = 3k^2$. Therefore the remainder when $n^2$ is divided by 3 is 0, so there does not exist an integer $k$ such that $n^2 = 3k + 2$, because $n^2$ is of the form $3k$, not $3k + 2$.

**Case 2:** Suppose $n = 3r + 1$ for some integer $r$. Then

$$n^2 = (3r + 1)^2 = 9r^2 + 6r + 1 = 3(3r^2 + 2r) + 1,$$

so $n^2 = 3t + 1$, where $t = 3r^2 + 2r$. Therefore the remainder when $n^2$ is divided by 3 is 1, so there does not exist an integer $k$ such that $n^2 = 3k + 2$, because $n^2$ is of the form $3k + 1$, not $3k + 2$.

**Case 3:** Suppose $n = 3r + 2$ for some integer $r$. Then

$$n^2 = (3r + 2)^2 = 9r^2 + 12r + 4 = 3(3r^2 + 4r + 1) + 1,$$

so $n^2 = 3u + 1$, where $u = 3r^2 + 4r + 1$. Therefore the remainder when $n^2$ is divided by 3 is 1, so there does not exist an integer $k$ such that $n^2 = 3k + 2$, because $n^2$ is of the form $3k + 1$, not $3k + 2$.

Thus, in all cases there does not exist an integer $k$ such that $n^2 = 3k + 2$, and the statement has been proved. □

**Problem 2.** (6 points) Let $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{3\}$, and $D = \{1, 2, 3\}$.

(a) (2 points) Use set builder notation to describe $D$.
(b) (4 points) Which of these sets are subsets of which other of these sets? In other words, list all true statements of the form $X \subseteq Y$, where $X$ and $Y$ are to be replaced with $A$, $B$, $C$, or $D$.

**Solution.**

(a) $D = \{\, x \in \mathbf{Z}^+ \mid x < 4 \,\}$. (Other answers are possible.)

(b) $A \subseteq A$, $A \subseteq D$, $B \subseteq B$, $B \subseteq D$, $C \subseteq B$, $C \subseteq C$, $C \subseteq D$, and $D \subseteq D$.

**Problem 3.** (9 points) Determine whether each of these statements is true or false.

(a) $0 \in \emptyset$
(b) $\emptyset \in \{0\}$
(c) $\{0\} \subseteq \emptyset$
(d) $\emptyset \subseteq \{0\}$
(e) $\{0\} \in \{0\}$
(f) $\{0\} \subseteq \{0\}$
(g) $\{\emptyset\} \subseteq \{\emptyset\}$
(h) $\emptyset \in \{\emptyset\}$
(i) $\emptyset \subseteq \{\emptyset\}$

**Solution.**

(a) False. The empty set $\emptyset$ contains no elements, so 0 cannot be an element of the empty set.

(b) False. The set $\{0\}$ contains only one element, which is the number 0. It does not contain the empty set as an element. Therefore, the empty set $\emptyset$ is not an element of the set $\{0\}$.

(c) False. The only subset of the empty set is the empty set, so the set $\{0\}$ is not a subset of the empty set. (The set $\{0\}$ contains an element, the number 0, which is not an element of the empty set; therefore $\{0\}$ is not a subset of the empty set.)

(d) True. The empty set $\emptyset$ is a subset of every set, so in particular it is a subset of the set $\{0\}$.

(e) False. The set $\{0\}$ contains exactly one element, which is the number 0. It does not contain the *set* $\{0\}$ as an element. It is important to see that the number 0 is not the same thing as the set $\{0\}$.

(f) True. Every set is a subset of itself, so in particular the set $\{0\}$ is a subset of itself.

(g) True. Again, every set is a subset of itself, so in particular the set $\{\emptyset\}$, which is the set containing the empty set $\emptyset$, is a subset of itself.

(h) True. The set $\{\emptyset\}$ is the set containing the empty set $\emptyset$, so the empty set is an element of the set $\{\emptyset\}$.

(i) True. The empty set $\emptyset$ is a subset of every set, so in particular it is a subset of the set $\{\emptyset\}$. Note that the reason this statement is true is **not** that the empty set is an element of the set $\{\emptyset\}$.

**Problem 4.** (5 points) Let $A$ and $B$ be finite sets. Let $m$ be the cardinality of $A$ and let $n$ be the cardinality of $B$. Find the cardinality of the following sets.

(a) $P(A)$
(b) $P(B)$
(c) $A \times B$
(d) $P(A \times B)$
(e) $P(A) \times P(B)$

**Solution.**

(a) If a set $S$ has $n$ elements, then its power set $P(S)$ has $2^n$ elements (see page 117 of the textbook). Therefore, since $A$ has $m$ elements, the power set $P(A)$ of $A$ has $2^m$ elements, so the cardinality of $P(A)$ is $2^m$.

(b) Since $B$ has $n$ elements, the power set $P(B)$ of $B$ has $2^n$ elements, so the cardinality of $P(B)$ is $2^n$.

(c) The set $A \times B$, called the *Cartesian product* of $A$ and $B$, consists of all ordered pairs of the form $(a, b)$, where $a$ is an element of $A$ and $b$ is an element of $B$. Since there are $m$ elements in $A$, we have $m$ different choices for the first element of an ordered pair of the form $(a, b)$; and since there are $n$ elements in $B$, for each of these $m$ different possibilities for the first element of the ordered pair we have $n$ different choices for the second element of the ordered pair. Altogether, then, we have $m \times n$ different ordered pairs of the form $(a, b)$. Therefore, the cardinality of $A \times B$ is $mn$.

(d) Since $A \times B$ has $mn$ elements, the power set $P(A \times B)$ of $A \times B$ has $2^{mn}$ elements, so the cardinality of $P(A \times B)$ is $2^{mn}$.

(e) The set $P(A) \times P(B)$ is the set of all ordered pairs of the form $(R, S)$, where $R$ is a subset of $A$ and $S$ is a subset of $B$. We know that there are $2^m$ different subsets of $A$ [from part (a) of this problem], and we know that there are $2^n$ different subsets of $B$ [from part (b)]. Using similar reasoning to that used in part (c), we see that there are $2^m \times 2^n$ different ordered pairs of the form $(R, S)$. Therefore, the cardinality of $P(A) \times P(B)$ is $2^m 2^n = 2^{m+n}$.

**Problem 5.** (8 points) Let

$$\begin{aligned} A &= \{1, 2, 3, 4, 5\}, \\ B &= \{\, x \mid x \text{ is even} \,\}, \\ C &= \{\, x \mid x \text{ is a multiple of } 3 \,\}, \text{ and} \\ D &= \{\, x \mid x \text{ is prime} \,\}, \end{aligned}$$

where the universal set is the set of all positive integers less than 20. [*Note*: The number 1 is not prime.]

(a) What is $A \cup C$?
(b) What is $B \cap C$?
(c) What is $A - D$?
(d) What is $D - A$?
(e) What is $\overline{A}$?
(f) What is $B \cap D$?
(g) What is $(B \cap C) \cap D$?
(h) What is $(B \cup D) \cap \overline{A}$?

**Solution.**

(a) The set $A \cup C$ consists of all elements that are in $A$ or in $C$, or in both, so

$$A \cup C = \{1, 2, 3, 4, 5, 6, 9, 12, 15, 18\}.$$

(b) The set $B \cap C$ consists of all elements that are in both $B$ and $C$, so it consists of all positive integers less than 20 that are both even and a multiple of 3. So we have

$$B \cap C = \{6, 12, 18\}.$$

This can also be written as

$$B \cap C = \{\, x \mid x \text{ is a multiple of } 6 \,\}.$$

(c) The set $A - D$ consists of all elements of $A$ that are not elements of $D$, so it contains all integers from 1 to 5 that are not prime, i.e.,

$$A - D = \{1, 4\}.$$

(d) The set $D - A$ consists of all elements of $D$ that are not elements of $A$, so it contains all prime numbers less than 20 that are not between 1 and 5, i.e.,

$$D - A = \{7, 11, 13, 17, 19\}.$$

(e) The set $\overline{A}$ consists of all elements that are not contained in $A$. Since our universal set is the set of all positive integers less than 20, we have

$$\overline{A} = \{6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19\} = \{6, 7, 8, \ldots, 19\}.$$

(f) The set $B \cap D$ consists of all elements that are in both $B$ and $D$, so it consists of all positive integers less than 20 that are both even and prime. There is only one such integer, so we have

$$B \cap D = \{2\}.$$

(g) The set $(B \cap C) \cap D$ consists of all elements that are in all three of the sets $B$, $C$, and $D$. Thus it contains all positive integers less than 20 that are even multiples of 3 and prime. Any number that is an even multiple of 3, however, must be divisible by both 2 and 3, so it cannot be prime. Therefore, there can be no elements in the set $(B \cap C) \cap D$, so we have

$$(B \cap C) \cap D = \emptyset.$$

(h) The set $(B \cup D) \cap \overline{A}$ consists of all elements that are either in $B$ or in $D$ (or both), and that additionally are in the complement of $A$. First, we figure out what $B \cup D$ is: this is the set of all positive integers less than 20 that are either even or prime, so

$$B \cup D = \{2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 19\}.$$

Now, the set $(B \cup D) \cap \overline{A}$ consists of all elements that are in both the set $B \cup D$ shown above and the set $\overline{A}$ shown in the answer to part (e). Hence,

$$(B \cup D) \cap \overline{A} = \{6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 19\}.$$

**Problem 6.** (8 points) List the ordered pairs in the relation $R$ from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$ where $(a, b) \in R$ if and only if

(a) $a = b$.
(b) $a + b = 4$.
(c) $a > b$.
(d) $a$ divides $b$.

**Solution.**

(a) An ordered pair $(a, b)$ is in $R$ if and only if $a = b$. Therefore, the ordered pairs in $R$ are $(0, 0)$, $(1, 1)$, $(2, 2)$, and $(3, 3)$. [Note that $(4, 4)$ is not in $R$, since $R$ is a relation from $A$ to $B$, and so the second element of the ordered pair must be an element of $B$; and $4 \notin B$.]

(b) An ordered pair $(a, b)$ is in $R$ if and only if $a + b = 4$. Therefore, the ordered pairs in $R$ are $(1, 3)$, $(2, 2)$, $(3, 1)$, and $(4, 0)$. [Note that $(0, 4)$ is not in $R$, since the second element of the ordered pair must be an element of $B$, and $4 \notin B$.]

(c) An ordered pair $(a, b)$ is in $R$ if and only if $a > b$. Therefore, the ordered pairs in $R$ are $(1, 0)$, $(2, 0)$, $(2, 1)$, $(3, 0)$, $(3, 1)$, $(3, 2)$, $(4, 0)$, $(4, 1)$, $(4, 2)$, and $(4, 3)$.

(d) An ordered pair $(a, b)$ is in $R$ if and only if $a$ divides $b$, that is, if and only if $b$ is a multiple of $a$. Therefore, the ordered pairs in $R$ are $(1, 0)$, $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 0)$, $(2, 2)$, $(3, 0)$, $(3, 3)$, and $(4, 0)$.

**Problem 7.** (16 points) For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is transitive, and whether it is an equivalence relation.

(a) $\{ (2,2), (2,3), (2,4), (3,2), (3,3), (3,4) \}$

(b) $\{ (1,1), (1,2), (2,1), (2,2), (3,3), (4,4) \}$

(c) $\{ (1,2), (2,3), (3,4) \}$

(d) $\{ (1,1), (1,3), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4) \}$

**Solution.**

(a) This relation is not reflexive, since it does not contain $(1, 1)$. It is not symmetric, since for example it contains $(2, 4)$ but not $(4, 2)$. However, it is transitive, because for all $a$, $b$, and $c$ in the set $\{1, 2, 3, 4\}$, if $(a, b)$ and $(b, c)$ are both in the relation, then $(a, c)$ is too; for example, $(2, 3)$ and $(3, 2)$ are both in the relation, and sure enough $(2, 2)$ is also in the relation. Since this relation is not reflexive and is not symmetric, it is not an equivalence relation.

(b) This relation is reflexive, because for all $a$ in the set $\{1, 2, 3, 4\}$, the ordered pair $(a, a)$ is in the relation; in other words, the relation contains all of $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$. It is also symmetric, because for all $a$ and $b$ in the set $\{1, 2, 3, 4\}$, if $(a, b)$ is in the relation, then $(b, a)$ is too; for example, $(1, 2)$ is in the relation, and $(2, 1)$ is too. This relation is also transitive. Since this relation is reflexive, symmetric, and transitive, it is an equivalence relation.

(c) This relation is not reflexive, since for example it does not contain $(1, 1)$. It is not symmetric, since for example it contains $(1, 2)$ but not $(2, 1)$. It is not transitive, since for example it contains $(1, 2)$ and $(2, 3)$, but not $(1, 3)$. Since this relation has none of these three properties, it is clearly not an equivalence relation.

(d) This relation is reflexive and symmetric, but it is not transitive, since for example it contains $(1, 3)$ and $(3, 2)$, but not $(1, 2)$. Therefore it is not an equivalence relation.

**Problem 8.** (10 points) Which of these relations on $\mathbf{Z}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

(a) $\{ (x, y) \mid x > y \}$
(b) $\{ (x, y) \mid x = y \}$
(c) $\{ (x, y) \mid |x - y| \leq 1 \}$
(d) $\{ (x, y) \mid x \text{ divides } y \}$
(e) $\{ (x, y) \mid |x| = |y| \}$

**Solution.**

(a) This relation is not an equivalence relation. It lacks reflexivity (since for example $5 \not> 5$) and symmetry (since for example $8 > 4$, but $4 \not> 8$). However, it is transitive, because if $a > b$ and $b > c$, then $a > c$.

(b) This relation is an equivalence relation. It is reflexive because $a = a$ for all $a$; it is symmetric because if $a = b$, then $b = a$; and it is transitive because if $a = b$ and $b = c$, then $a = c$.

(c) This relation is not an equivalence relation. It lacks transitivity, since for example $|3 - 2| \leq 1$ and $|2 - 1| \leq 1$ but $|3 - 1| > 1$. However, it is reflexive, since $|a - a| = 0 \leq 1$ for all $a$, and it is symmetric, since if $|a - b| \leq 1$, then $|b - a| \leq 1$ (this is easily seen by noting that $|a - b| = |b - a|$).

(d) This relation is not an equivalence relation. It lacks symmetry, since for example 5 divides 10, but not the other way round (10 does not divide 5). It is actually not reflexive either, since 0 does not divide 0 (though every other integer divides itself). However, it is transitive, because if $a$ divides $b$ and $b$ divides $c$, then $a$ divides $c$.

(e) This relation is an equivalence relation. It is reflexive because $|a| = |a|$ for all $a$; it is symmetric because if $|a| = |b|$, then $|b| = |a|$; and it is transitive, because if $|a| = |b|$ and $|b| = |c|$, then $|a| = |c|$.

**Problem 9.** (10 points) Find a partition of the positive integers such that no two prime numbers are in the same subset and every subset contains a prime number.

**Solution.** Let the first subset of the partition be $\{1, 2\}$, and for $i \geq 2$ let the $i$th subset of the partition consist of all integers greater than the $(i-1)$st prime number and less than or equal to the $i$th prime number. Then the partition is

$$\begin{aligned}\{ & \{1,2\}, \{3\}, \{4,5\}, \{6,7\}, \{8,9,10,11\}, \{12,13\}, \{14,15,16,17\}, \\ & \{18,19\}, \{20,21,22,23\}, \{24,25,26,27,28,29\}, \{30,31\}, \\ & \{32,33,34,35,36,37\}, \{38,39,40,41\}, \ldots \}.\end{aligned}$$

Alternatively, for $i \geq 1$ let the $i$th subset of the partition consist of all integers whose least prime factor is the $i$th prime number (and include 1 in the first subset). Then the partition is

$$\begin{aligned}\{ & \{1,2,4,6,8,10,12,14,16,18,20,\ldots\}, \\ & \{3,9,15,21,27,33,39,51,57,\ldots\}, \\ & \{5,25,35,55,65,85,95,\ldots\}, \\ & \{7,49,77,91,119,133,\ldots\}, \\ & \{11,121,143,187,209,\ldots\}, \\ & \{13,169,221,247,299,\ldots\}, \\ & \{17,289,323,391,493,\ldots\}, \ldots \}.\end{aligned}$$

Note that the first of these partitions has infinitely many subsets, each of which is finite, while the second has infinitely many subsets, each of which is infinite.