Improved Cubic Convolution for Two Dimensional Image Reconstruction

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ABSTRACT

This paper describes improved piecewise cubic convolution for two-dimensional image reconstruction. Piecewise cubic convolution is one of the most popular methods for image reconstruction, but the traditional approach uses a separable two-dimensional convolution kernel that is based on a one-dimensional derivation. The traditional approach is sub-optimal for the usual case of non-separable scenes and systems. The improved approach implements the most general two-dimensional, non-separable, piecewise cubic interpolator with constraints for symmetry, continuity, and smoothness.

1. INTRODUCTION

Image reconstruction is the process of defining a spatially continuous image from a set of discrete samples. It is fundamental to many digital image processing operations, such as translation, scaling, rotation, and geometric correction. These general operations require image values at locations for which no sample is available. Commonly, the values at arbitrary locations are estimated by a weighted average (or convolution) of the neighboring image samples. The weighting function used in local convolution is called the kernel. Convolution is used in Nearest-Neighbor, Bi-Linear, and Piecewise-Cubic reconstruction.

Piecewise cubic convolution has been used for image reconstruction since the 1970's [1]. The traditional piecewise cubic convolution kernel has been derived as a one-dimensional function and then generalized to two dimensions by assuming separability. Parametric Cubic Convolution (PCC) is a popular approach that imposes constraints to insure continuity and smoothness leaving one parameter that can be used to tune the kernel for the image [2]. Because PCC provides a good compromise between computational complexity and reconstruction accuracy, it is used widely in medical imaging and other applications.

Typical scenes and imaging systems are not separable, so the separable, two-dimensional piecewise cubic kernel is sub-optimal. The first reported non-separable piecewise cubic kernel for two-dimensional image reconstruction [3] yielded some improvements over the traditional method, but the approach was over-constrained to produce a two-parameter form and so it too is sub-optimal.

In this paper, we develop the most general non-separable, two-dimensional piecewise cubic convolution kernel defined on $[-2,2] \times [-2,2]$ with constraints for symmetry, continuity, and smoothness. This derivation has three parameters. We examine a Taylor series expansion of the error term under specific conditions to suggest image-independent values for the parameters.

2. TWO-DIMENSIONAL DERIVATION

2.1. Traditional Separable Derivation

It is useful to review the traditional one-dimensional derivation used in the separable kernel in order to introduce both concepts and notation. One-dimensional, piecewise-cubic reconstruction is implemented by convolving the samples of a digital image p with a piecewise-cubic kernel f to define the continuous result r:

$$r(x) = \sum_{m=-\infty}^{\infty} p[m] f(x-m) .$$

For notational convenience, the spatial coordinates are normalized in units of the sampling interval.

The kernel is constrained to the interval $[-2,2]$ and consists of cubic polynomial pieces connected at the knots at {-2,-1,0,1,2}. Symmetry requires:

$$\forall x, f(x) = f(-x) ,$$

so it is sufficient to define two pieces with a total of eight degrees of freedom:

$$f(x) = \begin{cases} a_3x^3 + a_2x^2 + a_1x + a_0 & 0 \le x \le 1 \\ b_3x^3 + b_2x^2 + b_1x + b_0 & 1 < x \le 2. \end{cases}$$

To insure continuous, smooth interpolation, it is necessary to impose constraints at the knots. To insure continuity,

$$\begin{aligned} \lim_{x \to 1-} f(x) &= \lim_{x \to 1+} f(x) \\ f(2) &= 0. \end{aligned}$$

Smoothness requires a continuous first-derivative:

$$\begin{aligned} f'(0) &= 0 \\ \lim_{x \to 1-} f'(x) &= \lim_{x \to 1+} f'(x) \\ f'(2) &= 0. \end{aligned}$$

Flat-field interpolation requires

$$\begin{aligned} f(0) &= 1 \\ \forall x, \sum_{m=-\infty}^{\infty} f(x-m) &= 1. \end{aligned}$$

These seven constraints leave only one degree of freedom, which can be identified with the slope of the kernel at $x = 1$. The resulting one-dimensional piecewise-cubic kernel is

$$f(x) = \begin{cases} (\alpha+2)x^3 - (\alpha+3)x^2 + 1 & 0 \le x \le 1 \\ \alpha x^3 - 5\alpha x^2 + 8\alpha x - 4\alpha & 1 < x \le 2. \end{cases}$$

where α is the first derivative or slope of the kernel at $x = 1$.

The kernel function can be written equivalently with a constant term and a term linear in α as

$$f(x) = f_0(x) + \alpha f_1(x)$$

where

$$\begin{aligned} f_0(x) &= \begin{cases} (x-1)(2x^2 - x - 1) & 0 \le x \le 1 \\ 0 & 1 < x \le 2 \end{cases} \\ f_1(x) &= \begin{cases} x^2(x-1) & 0 \le x \le 1 \\ x^3 - 5x^2 + 8x - 4 & 1 < x \le 2. \end{cases} \end{aligned}$$

The two-dimensional separable generalization of the one-dimensional piecewise cubic kernel is

$$f_s(x, y) = f(x) f(y).$$

However, the assumption of separability in this generalization to two dimensions is inconsistent with the fact that scenes and imaging systems are commonly not separable.

2.2. Non-Separable Derivation

The two-dimensional, biaxial symmetric, piecewise polynomial with cubic factors on the interval $[-2, 2] \times [-2, 2]$ is defined in the first quadrant by four two-dimensional pieces:

$$f(x, y) = \begin{cases} \sum_{j=0}^{3} \sum_{k=0}^{3} a_{jk} x^j y^k & \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{matrix} \\ \sum_{j=0}^{3} \sum_{k=0}^{3} b_{jk} x^j y^k & \begin{matrix} 1 < x \le 2 \\ 0 \le y \le 1 \end{matrix} \\ \sum_{j=0}^{3} \sum_{k=0}^{3} c_{jk} x^j y^k & \begin{matrix} 1 < x \le 2 \\ 1 < y \le 2 \end{matrix} \\ \sum_{j=0}^{3} \sum_{k=0}^{3} d_{jk} x^j y^k & \begin{matrix} 0 \le x \le 1 \\ 1 < y \le 2. \end{matrix} \end{cases}$$

This general symmetric form has 64 degrees of freedom.

For 90° rotational symmetry:

$$\forall (x, y), f(x, y) = f(y, x).$$

A more general (and complex) form would allow for axial and rotational asymmetries. For continuity between pieces:

$$\begin{aligned} \forall x, \lim_{y \to 1-} f(x, y) &= \lim_{y \to 1+} f(x, y) \\ \forall y, \lim_{x \to 1-} f(x, y) &= \lim_{x \to 1+} f(x, y) \\ \forall x, f(x, 2) &= 0 \\ \forall y, f(2, y) &= 0. \end{aligned}$$

For a continuous first-derivative between pieces:

$$\begin{aligned} \forall x, \left.\frac{\partial f}{\partial y}\right|_{(x,0)} &= 0 \\ \forall y, \left.\frac{\partial f}{\partial x}\right|_{(0,y)} &= 0 \\ \forall x, \lim_{y \to 1-} \left.\frac{\partial f}{\partial y}\right|_{(x,y)} &= \lim_{y \to 1+} \left.\frac{\partial f}{\partial y}\right|_{(x,y)} \\ \forall y, \lim_{x \to 1-} \left.\frac{\partial f}{\partial x}\right|_{(x,y)} &= \lim_{x \to 1+} \left.\frac{\partial f}{\partial x}\right|_{(x,y)} \\ \forall x, \left.\frac{\partial f}{\partial y}\right|_{(x,2)} &= 0 \\ \forall y, \left.\frac{\partial f}{\partial x}\right|_{(2,y)} &= 0. \end{aligned}$$

For flat-field interpolation:

$$\begin{aligned} f(0, 0) &= 1 \\ \forall (x, y), \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(x-m, y-n) &= 1. \end{aligned}$$

With these constraints, the non-separable form can be reduced to a function of three of the original coefficients as shown in Figure 1. The function can be rewritten in terms of the one parameter α of the separable filter, the second parameter β derived in [3], and a new parameter γ with the following substitutions:

$$\begin{aligned} a_{30} &= \alpha + 2 \\ a_{33} &= \beta + (\alpha+2)^2 \\ a_{32} &= \gamma - (\alpha+2)(\alpha+3) - \beta \end{aligned}$$

Then, the kernel can be expressed using the base functions of the separable kernel f_0 and f_1 and a new base function f_2:

$$\begin{aligned} f(x,y) &= (f_0(x) + \alpha f_1(x))(f_0(y) + \alpha f_1(y)) \\ &\quad + \beta f_1(x) f_1(y) + \gamma f_2(x,y). \end{aligned}$$

where

$$f_2(x,y) = \begin{cases} (x+y-2)x^2y^2 & 0 \le x \le 1 \\ & 0 \le y \le 1 \\ (4xy-3x-3y+2) & 0 \le x \le 1 \\ \quad \times y^2(x-2)^2 & 1 < y \le 2 \\ (8xy-7x-7y+6) & 1 < x \le 2 \\ \quad \times (x-2)^2(y-2)^2 & 1 < y \le 2 \\ (4xy-3x-3y+2) & 1 < x \le 2 \\ \quad \times x^2(y-2)^2 & 0 \le x \le 1. \end{cases}$$

The four components of the kernel (i.e., constant term, α term, β term, and γ term) are illustrated in Figure 2A-D.

3. IMAGE INDEPENDENT ANALYSIS

A Taylor series analysis of the one-dimensional reconstruction [4, 2] suggests the image-independent value $\alpha = -\frac{1}{2}$. This value for α provides third-order convergence at low frequencies as the sampling interval diminishes. Two-dimensional reconstruction with the new kernel can be analyzed in the same way to determine image-independent values for α, β, and γ.

If the image p is the result of sampling a scene s,

$$p(x,y) = s(x,y) \text{Ш}(x,y),$$

where Ш is a uniform lattice with Dirac-delta sampling impulses δ

$$\text{Ш}(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m, y-n),$$

then the expected or mean-square error (MSE) of sampling and reconstruction can be written as:

$$\epsilon^2 = E\left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |r(x,y) - s(x,y)|^2 \, dx\, dy \right\}.$$

As the sampling interval is narrowed to yield sufficient sampling, the MSE can be rewritten in the Fourier frequency domain as [2]:

$$\epsilon^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^2(u,v) \hat{\Phi}_s(u,v) \, dx\, dy$$

where (u,v) is the spatial frequency, $\hat{\Phi}_s$ is the expected power (or power spectrum) of the scene spectrum, and e^2 is an image-independent function that determines the MSE. In the limit at low frequencies, the Taylor series expansion for this error function is given in Figure 3.

In the Taylor series expansion, the (u^2+v^2) and (u^4+v^4) terms are zero if

$$\alpha = -\frac{1}{2} - 2\gamma. \tag{1}$$

In the one-dimensional analysis these terms reduce to zero if $\alpha = -\frac{1}{2}$, so if $\gamma = 0$ these terms are the same as for the traditional separable filter. With (1), the (u^2v^2) term reduces to zero if

$$\beta = -4\gamma^2 - 6\gamma. \tag{2}$$

With (1) and (2), the $(u^4v^2 + u^2v^4)$ term reduces to zero if

$$\gamma = 0 \quad \text{or} \quad \gamma = \tfrac{1}{6}. \tag{3}$$

Subsequent terms of the Taylor series do not reduce to zero for either value of γ.

4. CONCLUSIONS

This paper formulates the most general two-dimensional, non-separable, piecewise cubic interpolator with constraints for symmetry, continuity, and smoothness. The paper presents a Taylor series analysis of the image-independent reconstruction error at low frequencies with sufficient sampling, but much remains to be done to analyze performance.

5. REFERENCES

[1] Samuel S. Rifman, "Digital rectification of ERTS multispectral imagery," in *Proc. Symp. Significant Results Obtained from ERTS-1*. NASA SP-327, 1973, vol. I Sec. B, pp. 1131–1142.

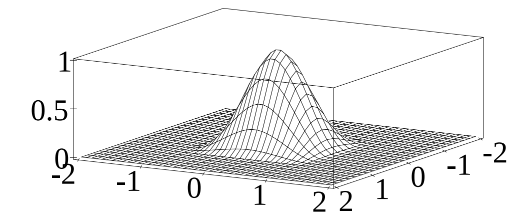
[2] Stephen K. Park and Robert A. Schowengerdt, "Image reconstruction by parametric cubic convolution," *Computer Vision, Graphics, and Image Processing*, vol. 23, pp. 258–272, 1983.

[3] Stephen E. Reichenbach and Frank Geng, "Image reconstruction with two-dimensional piecewise polynomial convolution," in *International Conference on Acoustics, Speech, and Signal Processing*. IEEE, 1999, pp. 3237–3240.

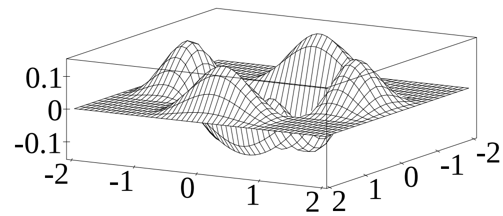
[4] Robert G. Keys, "Cubic convolution interpolation for digital image processing," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 29, no. 6, pp. 1153–1160, 1981.

$$f\left(x,y\right) = \begin{cases} \left(\left(x^3y^3 - x^2y^2\right)a_{33} + \left(x^3y^2 - 2x^2y^2 + x^2y^3\right)a_{32} & 0 \le x \le 1 \\ \qquad + \left(x^3 - x^2 - y^2 + y^3\right)a_{30} + \left(x^2y^2 - x^2 - y^2 + 1\right)\right) & 0 \le y \le 1 \\ \left(x-2\right)^2\left(\left(5xy^3 - 4y^3 - 4xy^2 + 3y^2\right)a_{33} + \left(4xy^3 - 3xy^2 - 3y^3 + 2y^2\right)a_{32} & 1 < x \le 2 \\ \qquad + \left(x - 2xy^2 + 2xy^3 + y^2 - 1 - y^3\right)a_{30} + \left(-2x + 2xy^2 + 2 - 2y^2\right)\right) & 0 \le y \le 1 \\ \left(y-2\right)^2\left(x-2\right)^2\left(\left(9xy - 8x - 8y + 7\right)a_{33} + \left(8xy - 7x - 7y + 6\right)a_{32} & 1 < x \le 2 \\ \qquad + \left(-3x + 4xy - 3y + 2\right)a_{30} + \left(4 + 4xy - 4x - 4y\right)\right) & 1 < y \le 2 \\ \left(y-2\right)^2\left(\left(5x^3y - 4x^2y + 3x^2 - 4x^3\right)a_{33} + \left(-3x^2y + 4x^3y + 2x^2 - 3x^3\right)a_{32} & 0 \le x \le 1 \\ \qquad + \left(y - 2x^2y + 2x^3y + x^2 - 1 - x^3\right)a_{30} + \left(2x^2y - 2y - 2x^2 + 2\right)\right) & 1 < y \le 2. \end{cases}$$

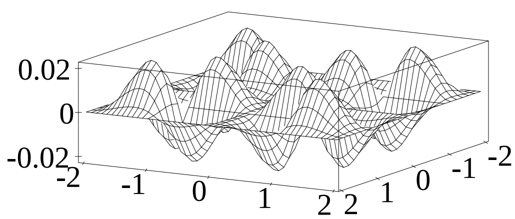
Fig. 1. The filter in terms of the original coefficients.



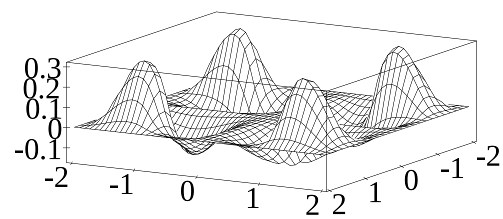
A. Constant term component, $f_0\left(x\right)f_0\left(y\right)$.



B. α term component, $f_0\left(x\right)f_1\left(y\right) + f_1\left(x\right)f_0\left(y\right)$.



C. α^2 and β terms component, $f_1\left(x\right)f_1\left(y\right)$.



D. γ term component, $f_2\left(x,y\right)$.

Fig. 2. Kernel components (spatial-domain left and Fourier-domain right).

$$\begin{aligned} e^2\left(u,v\right) &= \frac{2\pi^2}{105}\left(2\alpha+4\gamma+1\right)^2\left(u^2+v^2\right) + \frac{4\pi^4}{315}\left(14\alpha+28\gamma+9\right)\left(2\alpha+4\gamma+1\right)\left(u^4+v^4\right) + \frac{\pi^4}{7350}\left(337+1240\alpha\right. \\ &\quad -216\beta - 64\gamma + 920\alpha^2 + 16\beta^2 - 5248\gamma^2 - 416\alpha\beta - 480\alpha\gamma - 704\beta\gamma - 416\alpha^3 + 32\alpha^2\beta - 704\alpha^2\gamma \\ &\quad \left.+16\alpha^4\right)\left(u^2v^2\right) + \frac{4\pi^6}{33075}\left(-603 - 4752\alpha - 1376\beta - 35048\gamma - 8188\alpha^2 + 224\beta^2 - 91184\gamma^2 - 2080\alpha\beta\right. \\ &\quad \left.-71040\alpha\gamma + 128\beta\gamma - 2080\alpha^3 + 448\alpha^2\beta + 128\gamma\alpha^2 + 224\alpha^4\right)\left(u^4v^2 + u^2v^4\right) + \frac{4\pi^6}{4725}\left(23 + 484\alpha\right. \\ &\quad \left.+968\gamma + 836\alpha^2 + 3344\gamma^2 + 3344\alpha\gamma\right)\left(u^6+v^6\right) + \ldots \end{aligned}$$

Fig. 3. The Taylor series expansion of the error factor.