CUBIC CONVOLUTION FOR ONE-PASS RESTORATION AND RESAMPLING

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ABSTRACT

This paper describes an efficient filter that implements resampling and restoration of digital images in a single pass. The approach is to derive optimal parameters for a piecewise-cubic, reconstruction function based on a comprehensive model of the imaging process that accounts for acquisition blurring, sampling, noise, and reconstruction artifacts. For many data products produced from remotely sensed images, piecewise-cubic reconstruction is the standard algorithm for resampling. For these products, restoration with the optimal piecewise-cubic function requires virtually no additional processing resources and can significantly improve radiometric fidelity.

1. INTRODUCTION

This paper describes an efficient filter that resamples and restores digital images in a single pass. Resampling involves reconstruction, the process of determining image values at arbitrary spatial locations from the discrete pixels. Reconstruction is required in many imaging applications and is particularly important in remote sensing where images are resampled to correct for geometric distortion and to register, rescale, or otherwise remap. Restoration involves correcting for degradations introduced during the imaging process to obtain more accurate estimates of the scene radiance field. Restoration can yield significant improvements in the accuracy of radiometric measures by accounting for degradations introduced during image acquisition, including

- blurring related to atmospheric transmission, optical components, and the spatial integration of detectors;
- noise caused by the inherent variability of radiance fields, quantization, and various instrument phenomena; and
- artifacts related to aliasing of high spatial-frequency scene components in the sampled image.

In addition to these acquisition degradations, resampling usually is implemented with reconstruction methods that use a local weighted average (e.g., nearest-neighbor or bilinear interpolation) and, as such, involve additional blurring and sampling. Display devices also reconstruct a continuous radiance field by forming a display spot for each pixel — effectively blurring the discrete values to form a continuous image.

Traditional restoration methods have focused on blurring and noise, with the effects of sampling and reconstruction frequently not considered. As Schreiber writes, the effects of these processes "on the overall performance of systems is generally ignored in the literature, but is actually very large.[1, p. *vi*]" Optimal approaches to restoration must be based on a comprehensive system model that adequately accounts for sampling and reconstruction as well as blurring and noise.

The filter developed in this paper is a piecewise cubic reconstruction function. The piecewise cubic function is widely used in resampling to interpolate discrete pixel values, an operation commonly called cubic convolution or, in its one-parameter form, parametric cubic convolution (PCC). The piecewise cubic is efficient because it uses low-order polynomials and has a small region of spatial support. The piecewise cubic requires more computation than simpler nearest-neighbor and linear interpolation functions, but performs much better and can be parameterized for specific systems. For many data products produced from remotely sensed images, piecewise cubic reconstruction is the standard algorithm for resampling. For these products, restoration with the optimal piecewise cubic function requires virtually no additional processing resources and can significantly improve radiometric fidelity.

2. IMAGING SYSTEM MODEL

This paper employs the model of the basic components of the digital image acquisition process presented in [2]. By nature, the model is a simplification of the more complex interactions in real imaging systems, but the model captures the most fundamental effects of the acquisition process. The image acquisition process has three phases: image formation, sampling, and quantization. Mathematically, the process that produces

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the digital image p is modeled as:

$$p[n] = \int_{-\infty}^{\infty} h(n-x)\, s(x)\, dx + e[n] \quad (1)$$

where s is the scene radiance field, h is the image acquisition point-spread function (PSF), and e is error or noise. Pixels are indexed with integer coordinates $[n]$ and the continuous spatial coordinates (x) are normalized to the sampling interval. Function values are expressed on the digital number scale. This is a fairly modest model, but it is adequate to demonstrate the radiometric issues in resampling and to develop an improved restoration and resampling technique.

Image acquisition can be expressed equivalently in the spatial-frequency domain, where system functions are characterized by their transfer functions and instead of convolution we have the pointwise product. The equivalent frequency domain equation for $\hat{p}$, the Fourier transform of the image, is:

$$\hat{p}(v) = \sum_{\nu=-\infty}^{\infty} \hat{h}(v-\nu)\, \hat{s}(v-\nu) + \hat{e}(v) \quad (2)$$

where spatial frequencies (v) are normalized to the sampling frequency, $\hat{s}$ is spatial-frequency spectrum of the scene, $\hat{h}$ is the acquisition transfer function, and $\hat{e}$ is the spatial-frequency spectrum of the noise.

Convolution of the digital image p and the reconstruction PSF f is:

$$r(x) = \sum_{n=-\infty}^{\infty} f(x-n)\, p[n]. \quad (3)$$

The corresponding frequency-domain equation for reconstruction is

$$\hat{r}(v) = \hat{f}(v)\, \hat{p}(v) \quad (4)$$

where $\hat{f}$ is the reconstruction transfer function.

3. IMAGING SYSTEM ANALYSIS

There is a tradeoff in system design between errors related to blurring and errors related to aliasing. We begin by expressing quantitatively our goal to have the reconstructed image be as accurate a measure of the scene as possible. Linfoot[4] used the expected mean-square error between the scene s and the reconstructed image r

$$\begin{aligned} S^2 &= E\left\{\int_{-\infty}^{\infty} |s(x) - r(x)|^2\, dx\right\} \\ &= E\left\{\int_{-\infty}^{\infty} |\hat{s}(v) - \hat{r}(v)|^2\, dv\right\} \end{aligned} \quad (5)$$

to define image fidelity.

The expression for expected mean-square error can be written to make clearer the tradeoff between blurring and aliasing in system design:

$$\begin{aligned} S^2 = &\int_{-\infty}^{\infty} \Phi_s(v) \left|1 - \hat{h}(v)\hat{f}(v)\right|^2 dv \\ &+ \int_{-\infty}^{\infty} \left(\sum_{\nu=-\infty}^{\infty} \Phi_s(v-\nu) \left|\hat{h}(v-\nu)\right|^2\right) \left|\hat{f}(v)\right|^2 dv \\ &+ \int_{-\infty}^{\infty} \Phi_e(v) \left|\hat{f}(v)\right|^2 dv \end{aligned} \quad (6)$$

where Φ_s is the scene power spectrum and Φ_e is the noise power spectrum. This analysis assumes that the noise is signal-independent and that sidebands of the scene spectrum that alias to the same frequency are uncorrelated.

The first term represents the error associated with blurring the image by both the acquisition PSF h and the reconstruction PSF f. To minimize this term, $\hat{h}$ and $\hat{f}$ should be equal to one at all frequencies (which would mean that the system should not blur during acquisition nor during reconstruction). The second term represents the error associated with the aliasing. To minimize this term, $\hat{h}$ and $\hat{f}$ should be equal to zero at all frequencies (which would mean that the system would eliminate aliasing by not passing any signal). So, there is a clear tradeoff between blurring and aliasing. The final term is associated with system noise. To minimize this term, $\hat{f}$ should be equal to zero at all frequencies (which would mean that the system would eliminate noise by not passing any signal).

4. THE OPTIMAL PIECEWISE CUBIC

The optimal piecewise cubic maximizes radiometric fidelity (i.e., minimize the expected mean-square error in the resampled values relative to the scene radiance field) based on the end-to-end imaging system model presented in Section 2. The spatial support of the piecewise cubic function is between ± 2. A cubic polynomial is defined over each unit interval (with knots at 0, ± 1, and ± 2), so there are 16 degrees of freedom in the general form. To insure a continuous and smooth reconstruction, the following conditions are required for the function derivative and value at each knot:

$$\lim_{x \to k-} f'(x) = \lim_{x \to k+} f'(x) \quad (7)$$

$$\lim_{x \to k-} f(x) = \lim_{x \to k+} f(x) \quad (8)$$

for $k = 0, \pm 1, \pm 2$. This leaves six degrees of freedom that can be identified with the function slope and value at the internal knots:

$$\alpha_k = f'(k) \quad (9)$$

$$\beta_k = f(k) \quad (10)$$

for $k = 0$ and ± 1. To insure reconstruction does not change the mean, the integral of the cubic must be 1:

$$\int_{-\infty}^{\infty} f(x)\, dx \quad = \quad 1. \tag{11}$$

This constraint removes one of the degrees of freedom, which can be identified with the function value at 0:

$$\beta_0 = 1 - \beta_{-1} - \beta_1. \tag{12}$$

Separating components by parameter, the cubic is

$$f(x) = f_0(x) + \alpha_{-1} f_1(x+1) + \alpha_1 f_1(x-1) + \alpha_0 f_1(x) + \beta_{-1} f_2(1/2 - x) + \beta_1 f_2(x - 1/2) \tag{13}$$

with

$$f_0(x) = \begin{cases} 0 & \text{if } |x| \geq 1 \\ -2x^3 - 3x^2 + 1 & \text{if } -1 \leq x \leq 0 \\ 2x^3 - 3x^2 + 1 & \text{if } 0 \leq x \leq 1 \end{cases}$$

$$f_1(x) = \begin{cases} 0 & \text{if } |x| \geq 1 \\ x^3 + 2x^2 + x & \text{if } -1 \leq x \leq 0 \\ x^3 - 2x^2 + x & \text{if } 0 \leq x \leq 1 \end{cases}$$

$$f_2(x) = \begin{cases} 0 & \text{if } |x| \geq \frac{3}{2} \\ 2x^3 + 6x^2 + 9x/2 & \text{if } -\frac{3}{2} \leq x \leq -\frac{1}{2} \\ -4x^3 + 3x & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 2x^3 - 6x^2 + 9x/2 & \text{if } \frac{1}{2} \leq x \leq \frac{3}{2}. \end{cases}$$

With $\beta_{-1} = \beta_1 = 0$ and $\alpha_0 = 0$, these equations are identical to those in [3]. With $\beta_{-1} = \beta_1$ and $\alpha_0 = 0$, these equations are identical to those in [5].

The Fourier transform of this cubic is:

$$\hat{f}(v) = \hat{f}_0(v) + \alpha_{-1}\hat{f}_1(v)\, e^{i2\pi v} + \alpha_1 \hat{f}_1(v-1)\, e^{-i2\pi v} + \alpha_0 \hat{f}_1(v) + \beta_{-1}\hat{f}_2^*(v)\, e^{i\pi v} + \beta_1 \hat{f}_2(v)\, e^{-i\pi v} \tag{14}$$

with

$$\hat{f}_0(v) = \frac{3\sin^2(\pi v)}{\pi^4 v^4} - \frac{3\sin(2\pi v)}{2\pi^3 v^3}$$

$$\hat{f}_1(v) = i\left(\frac{1}{\pi^3 v^3} + \frac{\cos(2\pi v)}{2\pi^3 v^3} - \frac{3\sin(2\pi v)}{4\pi^4 v^4}\right)$$

$$\hat{f}_2(v) = i\left(\frac{-3\cos(\pi v)}{2\pi^3 v^3} + \frac{3\cos(3\pi v)}{2\pi^3 v^3} + \frac{6\sin^3(\pi v)}{\pi^4 v^4}\right).$$

The derivation that yields the solution for the optimal piecewise cubic can be expressed as a more general problem. To this end, we generalize the filter as:

$$\hat{f}(v) \quad = \quad \hat{f}_0(v) + \sum_{k=1}^{K} \gamma_k \hat{f}_k(v). \tag{15}$$

For the piecewise cubic function in Equation PCC, there are five parametric components (i.e., parameters α_{-1}, α_0, α_1, β_{-1}, and β_1) and so $K = 5$.

Then, in order to minimize the expected mean-square error, substitute this expression into Equation 6, take the derivative with respect to filter parameters γ_k, and set the result equal to zero. For real-valued imaging systems, this yields the system of linear equations:

$$\sum_{j=1}^{K} \gamma_j \Re\left(\int_{-\infty}^{\infty} \Phi_r(v)\, \hat{f}_j(v)\, \hat{f}_k^*(v)\, dv\right) = \Re\left(\int_{-\infty}^{\infty} \left(\Phi_{s,r}(v) - \Phi_r(v)\, \hat{f}_0(v)\right) \hat{f}_k^*(v)\, dv\right) \tag{16}$$

for $k = 1..K$. This is a system of K linear equations with K unknowns. Solving for the values of γ yields the optimal parameters.

5. CONCLUSION

This paper presents the development of an optimal piecewise cubic filter for restoration and resampling. The derivation minimizes the expected mean-square difference between the scene radiance field and the resulting image based on an end-to-end model of the imaging process. Our current work on this filter involves the determination of the optimal values for the planned Landsat 7 Enhanced Thematic Mapper (ETM+) and simulation experiments for the Landsat 7 imaging system using images from the Advanced Solid-State Array Spectroradiometer (ASAS).

6. ACKNOWLEDGEMENTS

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